ABSTRACT

General Moment Theorems with Applications

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A well-known result by Hardy-Ramanujan gives an asymptotic expression for the number of possible ways to write an integer as the sum of smaller integers. We first rederive this result in the framework of a saddle-point approximation using elementary properties of the Riemann zeta function. We then consider the general partitioning problem of writing an integer n as a sum of summands from a given sequence Λ of non-decreasing integers. Under suitable assumptions on the sequence Λ, the associated zeta function and saddle-point methods are used to obtain asymptotic results about the number of possibilities for such a partitioning. Higher moments of the sequence Λ as well as the expected number of summands and the variance are also calculated, then applications are made to various sequences, including those of Barnes and Epstein types. These results are of potential interest in statistical mechanics in the context of Bose-Einstein condensation.

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CHAPTER ONE

Introduction

While it[the theory of partitions] seems to have only little, or no practical application, it has, in a certain sense, just the right degree of difficulty: the problems are far from trivial–but, at the same time, they are not so hard as to discourage any attempt of a solution.

Emil Grosswald [13], 1966

The first significant work dealing with the theory of partitions can be attributed to Euler [8], published in 1748. Partition theory then led a somewhat quiet life until 1918, when Hardy and Ramanujan [14], using some very involved combinatorics, produced their celebrated theorem, which gives the asymptotic result for the number of ways to write an integer as the sum of lesser integers.

In the mid-1970's the area of asymptotic analysis blossomed with two of its primary works, those of Dingle [5] and Olver [21]. Among the plethora of information within these two books are some very nice results concerning the method of steepest descent, more commonly called the saddle-point method. Also at this time, a mass produced reprint of Andrews' book, *The Theory of Partitions* [1], became available, and L.B. Richmond wrote two papers [22, 23] using the saddle-point method to describe the moments of certain types of one-dimensional partitions. In the first of this two part series, *The Moments of Partitions I*, using asymptotic analysis instead of combinatorics, Richmond reproduces the results of Hardy-Ramanujan [14] as well as calculating higher moments and variance. In his second paper, *The Moments of Partitions II*, Richmond goes on to give asymptotic results for the moments of a general sequence whose associated zeta-function has only one singularity in the interval $(0, 1]$.

Though Richmond's results are remarkable, his use of the saddle-point method is not very clear. Also, he does not address the question of a sequence that gives rise to a zeta-function with arbitrarily many singularities at arbitrary values, a multidimensional zeta-function. Building upon the results of Richmond, and taking full advantage of the saddle-point method as well as asymptotic analysis, this thesis answers that question.

To introduce the layout of this thesis: Chapter 2 covers all of the zeta-function background needed. All of the results given can be found in the current literature. Background on the Riemmann zeta-function can be found in [7] and [30]. Though the Barnes and Epstein zeta-functions are a little more obscure, they are of paramount importance in mathematical physics, one need only look at the works of Barnes [3], Kirsten [18, 19], or Elizalde [6], for a full reckoning. Properties of a general zeta-function are addressed by Voros [31].

To emphasize the importance of the zeta-function of Riemann, we have entitled Chapter 3, *Riemann Type Moments*. Using our improved method, we recreate and correct the results of Richmond's first paper, among other things re-deriving the famous Hardy-Ramanujan theorem described above.

Chapter 4 brings us to the heart of this thesis. We improve Richmond's results by deriving what we call the General Moment Theorems. The first of these theorems addresses the question, given a sequence Λ, with limited assumptions, how many ways are there to write a natural number n as a sum of members from the sequence Λ? The other general moment theorem addresses higher moments. To conclude Chapter 4, we apply the General Moment Theorems to find the variance of the average number of summands.

In Chapter 5 we apply the General Moment Theorems to a variety of sequences, starting with the application of the General Moment Theorems to a sequence whose associated zeta-function has only one singularity. In essence, this is what Richmond describes in his second paper, except we assume no restriction on the location of the singularity. We then reproduce the results of Nanda [20] for 2- and 3-dimensional

Barnes-type partitions, and proceed to consider 2- and 3-dimensional Epstein-type partitions. Chapter 5 concludes with an application to partitions over eigenvalues of a partial differential equation, calculating the variance in each of the above cases.

The advent of mathematical physics, specifically the area of statistical mechanics in the context of Bose-Einstein condensation, has brought partition theory to the forefront of current research. With applications such as these we hope that the results of this thesis will be of interest to not only number theorists, but to the physics community as well.

CHAPTER TWO

Zeta-functions

If only I had the theorems! Then I should find the proofs easily enough.

G.B. Riemann[1]

2.1 Introduction

All of the results given in this chapter can be found in the current literature. Background of the Riemmann zeta-function can be found in [7] and [30]. Though the Barnes and Epstein zeta-functions are a little more obscure, they are of paramount importance in mathematical physics, one need only look at the works of Barnes [3], Kirsten [18, 19], or Elizalde [6], for a full reckoning. Properties of a general zeta-function are addressed by Voros [31].

2.2 Riemann Zeta-function

The zeta-function of Riemann is one of the most studied functions to date. Due to the large amount of information that has already been presented in other sources, we proceed to show only those results that will be used in our examination of partition theory. For further results see [7] or [30].

Definition 2.1. Let $s \in \mathbb{C}$ with $\Re s > 1$. We define the Riemann zeta-function as

$$\zeta_{\mathcal{R}}(s) = \sum_{n \in \mathbb{N}} \frac{1}{n^s}.$$

Definition 2.2. For $s \in \mathbb{C}$ with $\Re s > -1$, we define the Π-function as

$$\Pi(s) = \int_0^\infty t^s e^{-t} dt.$$

[1] Quoted in I Lakatos, Proofs and refutations (Cambridge, 1976)

The Π-function, while not used anymore, was the historical precursor to the Γ-function, which Riemann used. More specifically we have for all $s \in \mathbb{C}$, that

$$\Pi(s-1) = \Gamma(s),$$

so that they are equivalent. We will use this notation for the historical proof of the following proposition.

Proposition 2.1. *Let* $s \in \mathbb{C}$. *Then*

$$\zeta_{\mathcal{R}}(s) = \frac{\Pi(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x}, \tag{2.1}$$

where the limits of integration describe the contour $\mathcal{C}$*, which is shown in Figure 2.1.*

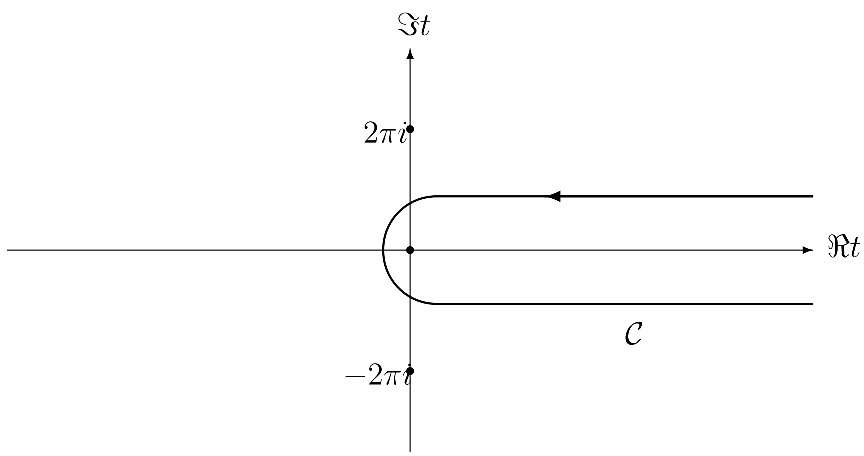

Figure 2.1. The Contour $\mathcal{C}$ in the Complex Plane.

Historical Proof. First we separate out the integral to

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} = \int_{\delta + i\varepsilon}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} + \int_{C^*(0,\delta)} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} + \int_{+\infty}^{\delta - i\varepsilon} \frac{(-x)^s}{e^x - 1} \frac{dx}{x},$$

where $C^*(0,\delta) = \delta e^{i\vartheta}$ with $\vartheta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$. By simple contour integration we have

$$\lim_{\delta \to 0} \frac{1}{2\pi i} \int_{C^*(0,\delta)} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} = 0$$

so that the middle integral disappears as $\delta \to 0$, and we have that

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} = \lim_{\delta \pm i\varepsilon \to 0} \left[\int_{\delta + i\varepsilon}^{+\infty} \frac{(x)^s(-1)^s}{e^x - 1} \frac{dx}{x} + \int_{+\infty}^{\delta - i\varepsilon} \frac{(x)^s(-1)^s}{e^x - 1} \frac{dx}{x} \right].$$

Note here that our contour contains a branch cut along the positive real axis, so that we must substitute carefully. Above the branch cut $e^{i\pi} = -1$, and below the branch cut $e^{-i\pi} = -1$ so that we have

$$\begin{aligned}\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} &= \lim_{\delta \pm i\varepsilon \to 0} \left[\int_{\delta + i\varepsilon}^{+\infty} \frac{(x)^s (e^{i\pi s})}{e^x - 1} \frac{dx}{x} - \int_{\delta - i\varepsilon}^{+\infty} \frac{(x)^s (e^{-i\pi s})}{e^x - 1} \frac{dx}{x} \right] \\ &= \lim_{\delta \to 0} \left[\int_{\delta}^{+\infty} \frac{(x)^s (e^{i\pi s} - e^{-i\pi s})}{e^x - 1} \frac{dx}{x} \right] \\ &= 2i \sin(\pi s) \int_0^{+\infty} x^{s-1} \sum_{n=1}^{\infty} e^{-nx} dx.\end{aligned}$$

Now we use the identity

$$\int_0^{+\infty} x^{s-1} \sum_{n=1}^{\infty} e^{-nx} dx = \int_0^{+\infty} \frac{(x)^{s-1}}{e^x - 1} dx = \Pi(s-1)\zeta_{\mathcal{R}}(s)$$

so that

$$\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} = 2i \sin(\pi s) \Pi(s-1) \zeta_{\mathcal{R}}(s).$$

Note that $\Pi(s) = s\Pi(s-1)$ so that [10] gives,

$$\sin(\pi s) = \frac{\pi s}{\Pi(s)\Pi(-s)} = \frac{\pi}{\Pi(s-1)\Pi(-s)}.$$

Then by substitution we have

$$\begin{aligned}\int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x} &= \frac{2\pi i \Pi(s-1)}{\Pi(s-1)\Pi(-s)} \zeta_{\mathcal{R}}(s) \\ &= \frac{2\pi i}{\Pi(-s)} \zeta_{\mathcal{R}}(s).\end{aligned}$$

With a little rearrangement we conclude

$$\zeta_{\mathcal{R}}(s) = \frac{\Pi(-s)}{2\pi i} \int_{+\infty}^{+\infty} \frac{(-x)^s}{e^x - 1} \frac{dx}{x},$$

which is the assertion. □

In the above proof we more or less showed that the two sides of the equation were equal. Though this is a valid proof, it is not very useful; that is, it is not constructive. For a more useful proof, we develop a construction involving the Γ-function defined below.

Definition 2.3. For $s \in \mathbb{C}$ we define the general Γ-function by

$$\Gamma(s) = \frac{-1}{2i\sin(\pi s)} \int_{\mathcal{C}} (-t)^{s-1} e^{-t} dt. \tag{2.2}$$

For $\Re s > 0$,

$$\Gamma(s) = \int_0^{\infty} t^{s-1} e^{-t} dt. \tag{2.3}$$

Lemma 2.1. *Let $\lambda \neq 0$ be a spectrum over a non-decreasing sequence of natural numbers, Λ. Then for $\Re s$ large enough*

$$\sum_{\lambda \in \Lambda} \frac{1}{\lambda^s} = \frac{i\Gamma(1-s)}{2\pi} \int_{\mathcal{C}} (-t)^{s-1} \sum_{\lambda \in \Lambda} e^{-\lambda t} dt. \tag{2.4}$$

Proof. First we write

$$\Gamma(s) = \frac{-1}{2i\sin(\pi s)} \int_{\mathcal{C}} (-x)^{s-1} e^{-x} dx.$$

Now let $x = \lambda t$ so that $dx = \lambda dt$. This substitution yields

$$\begin{aligned} \Gamma(s) &= \frac{-1}{2i\sin(\pi s)} \int_{\mathcal{C}} (-\lambda t)^{s-1} e^{-\lambda t} \lambda dt \\ &= \frac{-1}{2i\sin(\pi s)} \int_{\mathcal{C}} (-t)^{s-1} e^{-\lambda t} \lambda^s dt, \end{aligned}$$

so that dividing each side by λ^s and summing over Λ yields

$$\Gamma(s) \sum_{\lambda \in \Lambda} \frac{1}{\lambda^s} = \frac{-1}{2i\sin(\pi s)} \int_{\mathcal{C}} (-t)^{s-1} \sum_{\lambda \in \Lambda} e^{-\lambda t} dt.$$

Using the identity $2i\sin(\pi s) = \frac{2\pi i}{\Gamma(s)\Gamma(1-s)}$ gives

$$\sum_{\lambda \in \Lambda} \frac{1}{\lambda^s} = \frac{i\Gamma(1-s)}{2\pi} \int_{\mathcal{C}} (-t)^{s-1} \sum_{\lambda \in \Lambda} e^{-\lambda t} dt,$$

which is the assertion. □

Corollary 2.1. *Let $\lambda \neq 0$ be a spectrum over a sequence Λ, as described above. Then for $\Re s$ large enough*

$$\sum_{\lambda \in \Lambda} \frac{1}{\lambda^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \sum_{\lambda \in \Lambda} e^{-\lambda t} dt. \tag{2.5}$$

Constructive Proof of (2.1). In (2.4) we let $\lambda = n$ and let n range over $\mathbb{N}$, the natural numbers. This gives

$$\zeta_{\mathcal{R}}(s) = \sum_{n\in\mathbb{N}} \frac{1}{n^s} = \frac{i\Gamma(1-s)}{2\pi} \int_{\mathcal{C}} (-t)^{s-1} \sum_{n\in\mathbb{N}} e^{-nt} dt.$$

Note that $\sum_{n\in\mathbb{N}} e^{-nt} = \frac{1}{e^t-1}$ so that

$$\zeta_{\mathcal{R}}(s) = \frac{\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \frac{(-t)^s}{e^t-1} \frac{dt}{t},$$

which is the assertion. □

We continue in our analysis of $\zeta_{\mathcal{R}}(s)$ by computing values and residues of $\zeta_{\mathcal{R}}(s)$ at particular values of s. In this process we will see that the Riemann zeta-function is intimately connected to the Bernoulli numbers, which are defined as follows.

Definition 2.4. We define the nth Bernoulli number, B_n, by

$$\frac{t}{e^t-1} = \sum_{n\in\mathbb{N}_0} \frac{t^n}{n!} B_n. \tag{2.6}$$

This definition leads us to the following proposition.

Proposition 2.2. *Let* $n \in \mathbb{N}_0$. *Then*

$$\zeta_{\mathcal{R}}(-n) = (-1)^n \frac{B_{n+1}}{n+1}.$$

Proof. Using (2.1) we write

$$\begin{aligned} \zeta_{\mathcal{R}}(-n) &= \frac{\Gamma(n+1)}{2\pi i} \int_{\mathcal{C}} \frac{(-t)^{-n}}{e^t-1} \frac{dt}{t} \\ &= \frac{(-1)^n n!}{2\pi i} \int_{\mathcal{C}} t^{-n-2} \cdot \frac{t}{e^t-1} dt. \end{aligned}$$

Substituting with our definition for the Bernoulli numbers we have

$$\zeta_{\mathcal{R}}(-n) = \frac{(-1)^n n!}{2\pi i} \int_{\mathcal{C}} \sum_{m=0}^{\infty} \frac{B_m}{m!} t^{m-n-2} dt.$$

An easy application of the Cauchy Residue Theorem yields

$$\zeta_{\mathcal{R}}(-n) = (-1)^n \frac{B_{n+1}}{n+1},$$

which is our desired result. □

Let us now change our focus to the calculation of the residue of $\zeta_{\mathcal{R}}$. As a preliminary we compute the residues of the Γ-function as the following lemma.

Lemma 2.2. *The residues of* $\Gamma(s)$ *occur at* $s = -n$ *where* $n \in \mathbb{N}_0$, *and furthermore,*

$$\operatorname*{Res}_{s=-n} \{\Gamma(s)\} = \frac{(-1)^n}{n!}. \tag{2.7}$$

Proof. Note that (2.3) gives

$$\begin{aligned}\Gamma(s) &= \int_0^\infty t^{s-1} e^{-t} dt \\ &= \int_0^1 t^{s-1} e^{-t} dt + \int_1^\infty t^{s-1} e^{-t} dt.\end{aligned}$$

The right hand integral converges for all $s \in \mathbb{C}$ so that it does not contribute to the residues of $\Gamma(s)$.

Now we have

$$\int_0^1 t^{s-1} e^{-t} dt = \int_0^1 t^{s-1} \sum_{n \in \mathbb{N}_0} \frac{(-1)^n}{n!} t^n dt = \sum_{n \in \mathbb{N}_0} \frac{(-1)^n}{n!} \cdot \frac{1}{n+s},$$

and so the residues of $\Gamma(s)$ occur at $s = -n$ for $n \in \mathbb{N}_0$, and

$$\operatorname*{Res}_{s=-n} \{\Gamma(s)\} = \frac{(-1)^n}{n!}. \qquad \square$$

Using this we have the following proposition.

Proposition 2.3. *There is only one residue of* $\zeta_{\mathcal{R}}(s)$, *which occurs at* $s = 1$, *and*

$$\operatorname*{Res}_{s=1} \{\zeta_{\mathcal{R}}(s)\} = 1.$$

Proof. Starting with (2.5) and summing over $\mathbb{N}$ yields

$$\begin{aligned}\zeta_{\mathcal{R}}(s) &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{n \in \mathbb{N}} e^{-nt} dt \\ &= \frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{e^t - 1} dt + \frac{1}{\Gamma(s)} \int_1^\infty \frac{t^{s-1}}{e^t - 1} dt.\end{aligned}$$

Since $\Gamma(s)$ is analytic at $s = 1$ (i.e. $\Gamma(1) = 1$) and the right hand integral is convergent for all $s \in \mathbb{C}$, the second term will not contribute to the residue of $\zeta_{\mathcal{R}}$.

Taking a closer look at the integral of the first term, and using (2.6), gives

$$\int_0^1 \frac{t^{s-1}}{e^t - 1} dt = \int_0^1 t^{s-2} \sum_{n \in \mathbb{N}_0} \frac{t^n}{n!} B_n dt = \sum_{n \in \mathbb{N}_0} \frac{B_n}{n!} \cdot \frac{1}{n+s-1},$$

so that

$$\operatorname*{Res}_{s=1} \{\zeta_{\mathcal{R}}(s)\} = \frac{1}{\Gamma(1)} \cdot \frac{B_0}{0!} = 1.$$

As one looks at the above equality it is tempting to think that $\zeta_{\mathcal{R}}(s)$ will also have residues for all $-s \in \mathbb{N}$, because this would give $\frac{1}{n+s-1}$ singular behavior. This does not occur since $\frac{1}{\Gamma(s)} = 0$ for all $-s \in \mathbb{N}$ (see eq. (2.7)), and so the only residue of $\zeta_{\mathcal{R}}(s)$ occurs at $s = 1$. □

The final calculation we will make in this background section will be to calculate $\operatorname*{FP}_{s=1} \{\zeta_{\mathcal{R}}(s)\}$, the finite part of the Riemann zeta function, about $s = 1$. To do this we employ the functional equation for $\zeta_{\mathcal{R}}(s)$,

$$\zeta_{\mathcal{R}}(s) = 2^s \pi^{s-1} \sin\left(\frac{s\pi}{2}\right) \Gamma(1-s) \zeta_{\mathcal{R}}(1-s), \tag{2.8}$$

proof of which may be found in [7]. We now expand $\zeta_{\mathcal{R}}(s)$ about $s = 1 + \varepsilon$, via the functional equation to yield

$$\begin{aligned} \zeta_{\mathcal{R}}(1+\varepsilon) = &\left(2 + \varepsilon 2 \ln 2 + \mathcal{O}(\varepsilon^2)\right) \left(1 + \varepsilon \ln \pi + \mathcal{O}(\varepsilon^2)\right) \left(1 + \mathcal{O}(\varepsilon^2)\right) \times \\ &\times \operatorname*{Res}_{s=1} \{\Gamma(1-s)\} \left(1 - \varepsilon \psi(1) + \mathcal{O}(\varepsilon^2)\right) \left(\zeta_{\mathcal{R}}(0) - \varepsilon \zeta'_{\mathcal{R}}(0) + \mathcal{O}(\varepsilon^2)\right) \end{aligned} \tag{2.9}$$

where $\psi(z) = \frac{d}{dz} \ln \Gamma(z)$, is the Euler function. Note that [10] gives $\psi(1) = -\gamma$, where γ is Euler's number, and $\zeta'_{\mathcal{R}}(0) = -\frac{1}{2} \ln 2\pi$, so that one obtains the finite part of the Riemann zeta-function at $s = 1$ as

$$\operatorname*{FP}_{s=1} \{\zeta_{\mathcal{R}}(s)\} = -1(-\ln 2 - \ln \pi - \gamma + \ln 2\pi) = \gamma. \tag{2.10}$$

2.3 Barnes Zeta-function

After studying the zeta-function of Riemann, it is a logical step to generalize the theory to more general spectra. As a first example we look at the spectrum

$$\lambda = r_1 m_1 + r_2 m_2 + \cdots + r_d m_d = \vec{r} \vec{m},$$

where $r_i \in \mathbb{R}$ and $m_i \in \mathbb{N}_0$, for $i = 1, 2, \ldots, d$. This leads us to the zeta-function of Ernest William Barnes [3], commonly the Barnes zeta-function of dimension d, denoted $\zeta_{\mathcal{B}}$.

We begin our investigation of the Barnes zeta-function in the same spirit as the Riemann zeta-function, first deriving an integral representation, and then we continue by determining particular values and evaluating residues.

Definition 2.5. Let $s \in \mathbb{C}$ with $\Re s > d$, and $a \in \mathbb{R}$, $\vec{r} \in \mathbb{R}^d$ such that $a + \vec{m}\vec{r} > 0$ for all $\vec{m} \in \mathbb{N}_0^d$. We define the Barnes zeta function as

$$\zeta_{\mathcal{B}}(s, a|\vec{r}) = \sum_{\vec{m} \in \mathbb{N}_0^d} \frac{1}{(a + \vec{m}\vec{r})^s}.$$

Proposition 2.4. *Let $s \in \mathbb{C}$. Then*

$$\zeta_{\mathcal{B}}(s, a|\vec{r}) = \frac{i\Gamma(1-s)}{2\pi} \int_{\mathcal{C}} \frac{(-t)^{s-1} e^{-at}}{\prod_{j=1}^d (1 - e^{-r_j t})} dt. \tag{2.11}$$

Proof. In (2.4) we let $\lambda = a + \vec{m}\vec{r}$ and let $\vec{m}$ range over $\mathbb{N}_0^d$. This gives

$$\zeta_{\mathcal{B}}(s, a|\vec{r}) = \sum_{\vec{m} \in \mathbb{N}_0^d} \frac{1}{(a + \vec{m}\vec{r})^s} = \frac{i\Gamma(1-s)}{2\pi} \int_{\mathcal{C}} (-t)^{s-1} \sum_{\vec{m} \in \mathbb{N}_0^d} e^{-(a+\vec{m}\vec{r})t} dt,$$

which becomes

$$\zeta_{\mathcal{B}}(s, a|\vec{r}) = \frac{i\Gamma(1-s)}{2\pi} \int_{\mathcal{C}} \frac{(-t)^{s-1} e^{-at}}{\prod_{j=1}^d (1 - e^{-r_j t})} dt,$$

which is the proposition. □

Note that shrinking the contour to the non-negative real axis gives the following corollary.

Corollary 2.2. *For $\Re s > d$,*

$$\zeta_{\mathcal{B}}(s, a|\vec{r}) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{\prod_{j=1}^d (1 - e^{-r_j t})} dt. \tag{2.12}$$

We may view the Barnes zeta-function as a multidimensional Riemann zeta-function. Just as the Riemann zeta-function was intimately related to the Bernoulli numbers, there is an analogous relation between the Barnes zeta-function and the Bernoulli polynomials, in a way the multidimensional Bernoulli numbers. To see this relation we use the following definition, and proceed to find residues and values of $\zeta_{\mathcal{B}}$.

Definition 2.6. We define $B_n^{(d)}(a|\vec{r})$, the generalized d-dimensional Bernoulli polynomial, by

$$\frac{e^{-at}}{\prod_{j=1}^d (1-e^{-r_j t})} = \frac{(-1)^d}{\prod_{j=1}^d r_j} \sum_{n=0}^{\infty} \frac{(-t)^{n-d}}{n!} B_n^{(d)}(a|\vec{r}). \tag{2.13}$$

Proposition 2.5.

$$\operatorname*{Res}_{s=z} \{\zeta_{\mathcal{B}}(s,a|\vec{r})\} = \frac{(-1)^{d+z}}{(z-1)!(d-z)! \prod_{j=1}^d r_j} B_{d-z}^{(d)}(a|\vec{r}),$$

where $z = 1, \ldots, d$.

Proof. Let $z = 1, \ldots, d$, then we write,

$$\zeta_{\mathcal{B}}(s,a|\vec{r}) = \frac{-\Gamma(1-s)}{2\pi i} \int_{\mathcal{C}} \frac{(-t)^{s-1} e^{-at}}{\prod_{j=1}^d (1-e^{-r_j t})} dt.$$

Using our Bernoulli representation (2.13) at $s = z$, we yield

$$\begin{aligned} \frac{1}{2\pi i} \int_{\mathcal{C}} \frac{(-t)^{z-1} e^{-at}}{\prod_{j=1}^d (1-e^{-r_j t})} dt &= \operatorname*{Res}_{t=0} \left\{ \frac{(-t)^{z-1} e^{-at}}{\prod_{j=1}^d (1-e^{-r_j t})} \right\} \\ &= \frac{(-1)^{d-1}}{(d-z)! \prod_{j=1}^d r_j} B_{d-z}^{(d)}(a|\vec{r}). \end{aligned}$$

Substituting this back into our original equation and using the preceding Lemma, we have

$$\begin{aligned} \operatorname*{Res}_{s=z} \{\zeta_{\mathcal{B}}(s,a|\vec{r})\} &= \frac{(-1)^d}{(d-z)! \prod_{j=1}^d r_j} B_{d-z}^{(d)}(a|\vec{r}) \cdot \operatorname*{Res}_{s=z} \{\Gamma(1-s)\} \\ &= \frac{(-1)^{d+z}}{(z-1)!(d-z)! \prod_{j=1}^d r_j} B_{d-z}^{(d)}(a|\vec{r}), \end{aligned}$$

which is the assertion. □

Proposition 2.6. *Let $n \in \mathbb{N}_0$, then*

$$\zeta_{\mathcal{B}}(-n,a|\vec{r}) = \frac{(-1)^d n!}{(d+n)!\prod_{j=1}^d r_j} B_{d+n}^{(d)}(a|\vec{r}).$$

Proof. Let $n \in \mathbb{N}_0$, then we write,

$$\zeta_{\mathcal{B}}(-n,a|\vec{r}) = \frac{-\Gamma(1+n)}{2\pi i}\int_{\mathcal{C}} \frac{(-t)^{-n-1}e^{-at}}{\prod_{j=1}^d (1-e^{-r_j t})} dt.$$

Within the proof of the last proposition we gain

$$\frac{1}{2\pi i}\int_{\mathcal{C}} \frac{(-t)^{-n-1}e^{-at}}{\prod_{j=1}^d (1-e^{-r_j t})} dt = \frac{(-1)^{d-1}}{(d+n)!\prod_{j=1}^d r_j} B_{d+n}^{(d)}(a|\vec{r}),$$

so that since $\Gamma(1+n) = n!$ we yield the desired result

$$\zeta_{\mathcal{B}}(-n,a|\vec{r}) = \frac{(-1)^d n!}{(d+n)!\prod_{j=1}^d r_j} B_{d+n}^{(d)}(a|\vec{r}).$$

□

As we go further into our study of partition theory, we will be very interested in the specific case $\vec{r} = \vec{1}$. For $\vec{r} = \vec{1}$,

$$\zeta_{\mathcal{B}}(s,a) \equiv \zeta_{\mathcal{B}}(s,a|\vec{1}) = \sum_{m_i \in \mathbb{N}_0}^{\infty} (a + \sum_{i=1}^{d} m_i)^{-s} = \sum_{l=0}^{\infty} e_l^{(d)} (a+l)^{-s},$$

where $e_l^{(d)}$ is the number of ways to write l as a sum of at most d non-negative integers. Note that if $l=0$, then $e_l^{(d)} = 1$ since there is only one way to write 0 as the sum of non-negative integers. The first few values for $e_l^{(d)}$ are,

$$l=0, \quad \frac{1}{0!} = \begin{pmatrix} d-1 \\ d-1 \end{pmatrix}$$

$$l=1, \quad \frac{d}{1!} = \begin{pmatrix} d \\ d-1 \end{pmatrix}$$

$$l=2, \quad \frac{(d+1)d}{2!} = \begin{pmatrix} d+1 \\ d-1 \end{pmatrix}$$

$$l=3, \quad \frac{(d+2)(d+1)d}{3!} = \begin{pmatrix} d+2 \\ d-1 \end{pmatrix},$$

which motivate our next proposition.

Proposition 2.7. *Let $\vec{r} = \vec{1}$, then*

$$\zeta_{\mathcal{B}}(s,a) \equiv \zeta_{\mathcal{B}}(s,a|\vec{1}) = \sum_{l=0}^{\infty} e_l^{(d)} (a+l)^{-s},$$

where

$$e_l^{(d)} = \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix}.$$

This proposition is shown by proving the recurrence relationship provided above. It is summarized in the statement

$$\sum_{k=0}^{m} \begin{pmatrix} n+k \\ n \end{pmatrix} = \begin{pmatrix} n+m+1 \\ n+1 \end{pmatrix},$$

a proof of which may be found in most any undergraduate text on combinatorics, see [4] for details.

Ultimately we are striving to write the Barnes zeta-function as a linear combination of simpler zeta-functions; that is, to deal with the d dimensional spectrum as a combination of lesser dimensional spectra. To do this we need the following proposition.

Proposition 2.8. *For $e_l^{(d)}$ defined above, we have*

$$e_l^{(d)} = \sum_{i=0}^{d-1} g_i^{(d)}(a)(l+a)^{d-1-i},$$

where $g_i^{(d)}(a) = \frac{(-1)^i}{(d-i-1)!i!} B_i^{(d)}(a)$.

Proof. Let us define $g_i^{(d)}(a)$ through the equation

$$e_l^{(d)} = \begin{pmatrix} l+d-1 \\ d-1 \end{pmatrix} = \sum_{i=0}^{d-1} g_i^{(d)}(a)(l+a)^{d-1-i}.$$

Differentiating we get that

$$\frac{d}{da}e_l^{(d)} = 0 = \sum_{i=0}^{d-1}\left[(a+l)^{d-1-i}\frac{d}{da}g_i^{(d)}(a) + (d-1-i)g_i^{(d)}(a)(a+l)^{d-1-(i+1)}\right]$$
$$= (a+l)^{d-1}\frac{d}{da}g_0^{(d)}(a) +$$
$$+\sum_{i=0}^{d-1}\left[(d-(i+1))g_i^{(d)}(a) + \frac{d}{da}g_{i+1}^{(d)}(a)\right](a+l)^{d-1-(i+1)}$$

so that

$$\frac{d}{da}g_0^{(d)}(a) = 0$$

and

$$\frac{d}{da}g_{i+1}^{(d)}(a) = (i+1-d)g_i^{(d)}(a),$$

for $i = 0, 2, \ldots, d-1$. Note that the Bernoulli polynomials satisfy the relationship

$$\frac{d}{da}B_{i+1}^{(d)}(a) = (i+1)B_i^{(d)}(a),$$

so that a natural ansatz is

$$g_i^{(d)}(a) = C(i,d)B_i^{(d)}(a),$$

where $C(i,d)$ is some constant depending on i and d. Some calculation reveals that $C(0,d) = \frac{1}{(d-1)!}$, so that recursively we have $g_i^{(d)}(a) = \frac{(-1)^i}{(d-i-1)!i!}B_i^{(d)}(a)$. □

At this point a relationship between ζ_H, the Hurwitz zeta-function (defined below), and $\zeta_{\mathcal{B}}$ is established. This will be useful later, when computing finite parts of $\zeta_{\mathcal{B}}$, which arise in Section 4.2.2, when discussing higher general moments.

Definition 2.7. Let $s \in \mathbb{C}$ with $\Re s > 1$ and $a \in \mathbb{R}$ with $a > 0$. We define the Hurwitz zeta function as

$$\zeta_H(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s}.$$

Corollary 2.3. *Let $s \in \mathbb{C}$ with $\Re s > d$ and $a \in \mathbb{R}$ with $a > 0$.*

$$\zeta_{\mathcal{B}}(s,a) = \sum_{k=1}^{d}\frac{(-1)^{k+d}}{(k-1)!(d-k)!}B_{d-k}^{(d)}(a)\zeta_H(s+1-k,a) \tag{2.14}$$

where $B_i^{(d)}(a) = B_i^{(d)}(a|\vec{1})$.

Proof. By calculation we have

$$\begin{aligned}\zeta_{\mathcal{B}}(s,a) &= \sum_{l=0}^{\infty} e_l^{(d)}(a+l)^{-s} \\ &= \sum_{l=0}^{\infty}\sum_{i=0}^{d-1} g_i^{(d)}(a)(a+l)^{d-1-i-s} \\ &= \sum_{i=0}^{d-1} g_i^{(d)}(a)\zeta_H(s+1+i-d,a),\end{aligned}$$

where $g_i^{(d)}(a)$ is defined as in the previous proposition. Using this

$$\zeta_{\mathcal{B}}(s,a) = \sum_{i=0}^{d-1} \frac{(-1)^i}{(d-i-1)!i!} B_i^{(d)}(a)\zeta_H(s+1+i-d,a).$$

Now we set $i = d-k$ and run k from 1 to d, so that

$$\zeta_{\mathcal{B}}(s,a) = \sum_{k=1}^{d} \frac{(-1)^{k+d}}{(k-1)!(d-k)!} B_{d-k}^{(d)}(a)\zeta_H(s+1-k,a),$$

which is the desired result. □

The only finite part of the Barnes zeta-function that arises later is that of the singularity at the dimension. That is for a d-dimensional Barnes zeta-function, $\zeta_{\mathcal{B}}^{(d)}$, we need only determine $\operatorname*{FP}_{s=d}\{\zeta_{\mathcal{B}}(s,a|\vec{r})\}$ for the results of this paper. We proceed as follows. We first write, using (2.12),

$$\begin{aligned}\zeta_{\mathcal{B}}^{(d)}(s,a|\vec{r}) &= \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{t^{s-1}e^{-at}}{\prod_{j=1}^{d}(e^{r_j t}-1)}dt \\ &= \frac{1}{\Gamma(s)}\int_0^{\infty} \frac{t^{s-1}e^{-\left(a+\sum_{j=1}^{d} r_j\right)t}}{\prod_{j=1}^{d}(1-e^{-r_j t})}dt.\end{aligned}$$

The above integral contains the pole at $t = d$. Note that as we approach our lower integration bound, as $t \to 0$, and as $s \to d$ the integrand behaves like $\frac{1}{t}$, and so the integral is divergent at $s = d$. We wish to separate out the divergent part of the integral at $s = d$, thus we add and subtract the asymptotics as $t \to 0$. We continue the above equalities, as

$$\begin{aligned}\zeta_{\mathcal{B}}^{(d)}(s,a|\vec{r}) &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\left(a+\sum_{j=1}^d r_j\right)t}\times \\ &\qquad \times\left[\left(\prod_{j=1}^d \frac{1}{(1-e^{-r_j t})}\right) - \frac{1}{\left(\prod_{j=1}^d r_j\right)t^d} + \frac{1}{\left(\prod_{j=1}^d r_j\right)t^d}\right]dt \\ &= \frac{\Gamma(s-d)}{\Gamma(s)\left(\prod_{j=1}^d r_j\right)}\left(a+\sum_{j=1}^d r_j\right)^d + \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-\left(a+\sum_{j=1}^d r_j\right)t}\times \\ &\qquad \times\left[\left(\prod_{j=1}^d \frac{1}{(1-e^{-r_j t})}\right) - \frac{1}{\left(\prod_{j=1}^d r_j\right)t^d}\right]dt.\end{aligned}$$

We denote the second term of the above sum by $I_{\mathcal{B}}^{(d)}(s,a|\vec{r})$. The integrand of $I_{\mathcal{B}}^{(d)}(s,a|\vec{r})$ behaves like $\frac{1}{t^2}$ as $t \to 0$ and $s \to d$, so that $I_{\mathcal{B}}^{(d)}(s,a|\vec{r})$ is convergent at $s = d$, thus the pole at $s = d$ of $\zeta_{\mathcal{B}}^{(d)}(s,a|\vec{r})$ is contained in the first term. Expanding the first term about $s = d + \varepsilon$ gives

$$\begin{aligned}\frac{\Gamma(\varepsilon)}{\Gamma(d+\varepsilon)\left(\prod_{j=1}^d r_j\right)}\left(a+\sum_{j=1}^d r_j\right)^d &= \left(a+\sum_{j=1}^d r_j\right)^d \frac{1}{\varepsilon}\operatorname*{Res}_{s=d}\{\Gamma(s-d)\}\frac{1}{\Gamma(d)}\times \\ &\qquad \times\left(1-\varepsilon\gamma+\mathcal{O}(\varepsilon^2)\right)\left(1-\varepsilon\psi(d)+\mathcal{O}(\varepsilon^2)\right),\end{aligned}$$

where $\psi(z) = \frac{d}{dz}\ln\Gamma(z)$, is the Euler function, and γ the Euler number. Thus we conclude with the following proposition.

Proposition 2.9. *The finite part of the d-dimensional Barnes zeta-function at $s = d$ is*

$$\operatorname*{FP}_{s=d}\left\{\zeta_{\mathcal{B}}^{(d)}(s,a|\vec{r})\right\} = -\left(a+\sum_{j=1}^d r_j\right)^d\left(\frac{\psi(d)+\gamma}{\Gamma(d)}\right) + I_{\mathcal{B}}^{(d)}(d,a|\vec{r}) \qquad (2.15)$$

where $I_{\mathcal{B}}^{(d)}(s,a|\vec{r})$ is defined as above.

2.4 Epstein Zeta-function

As a preliminary to looking at the general Epstein zeta-function, we will look at the small t behavior of the sum

$$\sum_n e^{-an^2t}$$

where $a > 0$ and we are summing over $\mathbb{Z}, \mathbb{N}_0$, or $\mathbb{N}$. Note that with use of Fourier Transforms (see Appendix B) we have

$$\sum_{n\in\mathbb{Z}} e^{-n^2\pi x} = \frac{1}{\sqrt{x}} \sum_{n\in\mathbb{Z}} e^{-n^2\pi/x}.$$

Making the substitution $x = \frac{at}{\pi}$ we gain

$$\sum_{n\in\mathbb{Z}} e^{-an^2t} = \sqrt{\frac{\pi}{at}} \sum_{n\in\mathbb{Z}} e^{-n^2\pi^2/at}. \tag{2.16}$$

Focussing on the right most sum in (2.16) we see that

$$\sum_{n\in\mathbb{Z}} e^{-n^2\pi^2/at} = 1 + \mathcal{O}(e^{-\pi^2/at}),$$

so that

$$\sum_{n\in\mathbb{Z}} e^{-an^2t} = \sqrt{\frac{\pi}{at}} \left[1 + \mathcal{O}(e^{-\pi^2/at})\right]. \tag{2.17}$$

From now on we will use "$\sim$", to denote equality up to exponentially damped terms. We now apply this to three different summation ranges which result from the eigenvalues of $-\frac{\partial^2}{\partial x^2}$ on the interval with periodic, Neumann, and Dirichlet boundary conditions.

For the periodic boundary condition we are interested in the sum over $\mathbb{Z}$. From the above calculations it is quite immediate that for small t values

$$\sum_{n\in\mathbb{Z}} e^{-an^2t} \sim \sqrt{\frac{\pi}{at}}. \tag{2.18}$$

For the Neumann boundary condition we are interested in the sum over $\mathbb{N}_0$. Note that using (2.16) we have

$$\begin{aligned}\sum_{n\in\mathbb{N}_0} e^{-an^2t} &= \frac{1}{2}\left(\sum_{n\in\mathbb{Z}} e^{-an^2t}\right) + \frac{1}{2} \\ &= \frac{1}{2}\left(\sqrt{\frac{\pi}{at}}\sum_{n\in\mathbb{Z}} e^{-n^2\pi^2/at}\right) + \frac{1}{2}\end{aligned}$$

so that

$$\sum_{n\in\mathbb{N}_0} e^{-an^2t} \sim \frac{1}{2}\sqrt{\frac{\pi}{at}} + \frac{1}{2}. \tag{2.19}$$

Likewise for the Dirichlet boundary condition we are interested in the sum over $\mathbb{N}$. Note that the Dirichlet case is just a shift of the Neumann case ($\mathbb{N}_0 \to \mathbb{N}$), so that we merely subtract 1, which gives

$$\sum_{n\in\mathbb{N}} e^{-an^2t} \sim \frac{1}{2}\sqrt{\frac{\pi}{at}} - \frac{1}{2}. \tag{2.20}$$

It is also worthy to note that the small t behavior of the above series depend heavily on (2.17).

To continue on in the same fashion, let us consider the small t behavior of the series

$$\sum_{n_1,n_2} e^{-a_1n_1^2t-a_2n_2^2t}$$

where $a_1, a_2 > 0$ and for each index we are summing over $\mathbb{Z}, \mathbb{N}_0$, or $\mathbb{N}$. When summing over $\mathbb{Z}$ we have

$$\sum_{n_1,n_2\in\mathbb{Z}} e^{-a_1n_1^2t-a_2n_2^2t} = \left(\sum_{n_1\in\mathbb{Z}} e^{-a_1n_1^2t}\right)\left(\sum_{n_2\in\mathbb{Z}} e^{-a_2n_2^2t}\right)$$

so that (2.18) gives

$$\sum_{n_1,n_2\in\mathbb{Z}} e^{-a_1n_1^2t-a_2n_2^2t} \sim \frac{\pi}{\sqrt{a_1a_2}}t^{-1}.$$

Likewise summing over $\mathbb{N}_0$ we have

$$\sum_{n_1,n_2\in\mathbb{N}_0} e^{-a_1n_1^2t-a_2n_2^2t} = \left(\sum_{n_1\in\mathbb{N}_0} e^{-a_1n_1^2t}\right)\left(\sum_{n_2\in\mathbb{N}_0} e^{-a_2n_2^2t}\right)$$

so that (2.19) gives

$$\sum_{n_1,n_2\in\mathbb{N}_0} e^{-a_1n_1^2t-a_2n_2^2t} \sim \frac{1}{2^2}\left(\sqrt{\frac{\pi}{a_1t}}+1\right)\left(\sqrt{\frac{\pi}{a_2t}}+1\right)$$
$$= \frac{\pi}{4\sqrt{a_1a_2}}t^{-1}+\frac{1}{4}\left(\sqrt{\frac{\pi}{a_1}}+\sqrt{\frac{\pi}{a_2}}\right)t^{-\frac{1}{2}}+\frac{1}{4}.$$

Finally, with the Dirichlet boundary condition in mind, summing over $\mathbb{N}$ we have

$$\sum_{n_1,n_2\in\mathbb{N}} e^{-a_1n_1^2t-a_2n_2^2t} = \left(\sum_{n_1\in\mathbb{N}} e^{-a_1n_1^2t}\right)\left(\sum_{n_2\in\mathbb{N}} e^{-a_2n_2^2t}\right)$$

so that (2.20) gives

$$\sum_{n_1,n_2\in\mathbb{N}} e^{-a_1n_1^2t-a_2n_2^2t} \sim \frac{1}{2^2}\left(\sqrt{\frac{\pi}{a_1t}}-1\right)\left(\sqrt{\frac{\pi}{a_2t}}-1\right)$$
$$= \frac{\pi}{4\sqrt{a_1a_2}}t^{-1}-\frac{1}{4}\left(\sqrt{\frac{\pi}{a_1}}+\sqrt{\frac{\pi}{a_2}}\right)t^{-\frac{1}{2}}+\frac{1}{4}.$$

Again note that all of the above relations depend on (2.16) and (2.17).

Finally we look at the small t behavior of the general case. Let $\vec{n}=(n_1,n_2,\ldots,n_d)$, and consider the series

$$\sum_{\vec{n}} e^{-a_1n_1^2t-a_2n_2^2t-\cdots-a_dn_d^2t}$$

where $a_k>0$ and $k=1,2,\ldots d$, and we are summing $\vec{n}$ over $\mathbb{Z}^d,\mathbb{N}_0^d$, or $\mathbb{N}^d$. When summing over $\mathbb{Z}^d$ we have

$$\sum_{\vec{n}\in\mathbb{Z}^d} e^{-a_1n_1^2t-a_2n_2^2t-\cdots-a_dn_d^2t} = \prod_{k=1}^{d}\left(\sum_{n_k\in\mathbb{Z}} e^{-a_kn_k^2t}\right)$$
$$= \prod_{k=1}^{d}\left(\sqrt{\frac{\pi}{a_kt}}\sum_{n_k\in\mathbb{Z}} e^{-n_k^2\pi^2/a_kt}\right)$$

so that (2.18) gives

$$\sum_{\vec{n}\in\mathbb{Z}^d} e^{-a_1n_1^2t-a_2n_2^2t-\cdots-a_dn_d^2t} \sim \prod_{k=1}^{d}\sqrt{\frac{\pi}{a_kt}} = \frac{\pi^{\frac{d}{2}}}{\sqrt{a_1a_2\cdots a_d}}t^{-\frac{d}{2}}.$$

Likewise summing over $\mathbb{N}_0^d$ we have

$$\sum_{\vec{n}\in\mathbb{N}_0^d} e^{-a_1 n_1^2 t - a_2 n_2^2 t - \cdots - a_d n_d^2 t} = \prod_{k=1}^{d}\left(\sum_{n_1\in\mathbb{N}_0} e^{-a_1 n_1^2 t}\right)$$

so that (2.19) gives

$$\sum_{\vec{n}\in\mathbb{N}_0^d} e^{-a_1 n_1^2 t - a_2 n_2^2 t - \cdots - a_d n_d^2 t} \sim \frac{1}{2^d}\prod_{k=1}^{d}\left(\sqrt{\frac{\pi}{a_k t}} + 1\right). \tag{2.21}$$

It is convenient here to transform the above finite product into a finite sum; that is,

$$\prod_{k=1}^{d}\left(\sqrt{\frac{\pi}{r_k t}} + 1\right) = \sum_{n=0}^{d} c_n t^{\frac{-n}{2}} \tag{2.22}$$

where $c_0 = 1$, and for $n = 1, 2, \ldots, d$,

$$c_n = \pi^{\frac{k}{2}} \sum_{\substack{|v|=n \\ v\in\mathcal{P}(\{1,2,\ldots,d\})}} \left(\prod_{i\in v}\frac{1}{\sqrt{r_i}}\right), \tag{2.23}$$

where $\mathcal{P}(\{1,2,\ldots,d\})$ is the power set of $\{1,2,\ldots,d\}$ (i.e. the set of all subsets of $\{1,2,\ldots,d\}$). Thus we have

$$\sum_{\vec{n}\in\mathbb{N}_0^d} e^{-a_1 n_1^2 t - a_2 n_2^2 t - \cdots - a_d n_d^2 t} \sim \frac{1}{2^d}\prod_{k=1}^{d}\left(\sqrt{\frac{\pi}{a_k t}} + 1\right) = \frac{1}{2^d}\sum_{n=0}^{d} c_n t^{\frac{-n}{2}},$$

where the c_n's are described above.

Finally, with the Dirichlet boundary condition in mind, summing over $\mathbb{N}^d$ we have

$$\sum_{\vec{n}\in\mathbb{N}^d} e^{-a_1 n_1^2 t - a_2 n_2^2 t - \cdots - a_d n_d^2 t} = \prod_{k=1}^{d}\left(\sum_{n_k\in\mathbb{N}} e^{-a_k n_k^2 t}\right)$$

so that (2.20) gives

$$\sum_{\vec{n}\in\mathbb{N}^d} e^{-a_1 n_1^2 t - a_2 n_2^2 t - \cdots - a_d n_d^2 t} \sim \frac{1}{2^d}\prod_{k=1}^{d}\left(\sqrt{\frac{\pi}{a_k t}} - 1\right).$$

As in the proceeding case transforming the above finite product into a finite sum, using the same c_n's we have

$$\prod_{k=1}^{d}\left(\sqrt{\frac{\pi}{r_k t}} - 1\right) = \sum_{n=0}^{d}(-1)^{d-n} c_n t^{\frac{-n}{2}}$$

and so

$$\sum_{\vec{n}\in\mathbb{N}_0^d} e^{-a_1 n_1^2 t - a_2 n_2^2 t - \cdots - a_d n_d^2 t} \sim \frac{1}{2^d}\prod_{k=1}^{d}\left(\sqrt{\frac{\pi}{a_k t}}-1\right) = \frac{1}{2^d}\sum_{n=0}^{d}(-1)^{d-n} c_n t^{\frac{-n}{2}}.$$

To start our analysis of the Epstein zeta function, let us recall the following definitions.

Definition 2.8. Define $Q(\vec{m},\vec{r}) = r_1 m_1^2 + r_2 m_2^2 + \cdots + r_d m_d^2$. Let $s \in \mathbb{C}$ with $\Re s > \frac{d}{2}$, and $a \in \mathbb{R}$, $\vec{r} \in \mathbb{R}^d$ such that $a + Q(\vec{m},\vec{r}) > 0$ for all $\vec{m} \in \mathbb{N}_0^d$. We define the Epstein zeta function as

$$\zeta_{\mathcal{E}}(s,a|\vec{r}) = \sum_{\vec{m}\in\mathbb{N}_0^d} \frac{1}{(a+Q(\vec{m},\vec{r}))^s}.$$

Using the Γ-function as we have with other zeta functions we come up with the following representation:

Proposition 2.10. *Let $s \in \mathbb{C}$, then*

$$\zeta_{\mathcal{E}}(s,a|\vec{r}) = \frac{i\Gamma(1-s)}{2\pi}\int_{\mathcal{C}}(-t)^{s-1}e^{-at}\sum_{\vec{m}\in\mathbb{N}_0^d} e^{-Q(\vec{m},\vec{r})t}dt,$$

where $Q(\vec{m},\vec{r})$ is defined as above.

Proof. In (2.4) we let $\lambda = a + Q(\vec{m},\vec{r})$ and let $\vec{m}$ range over $\mathbb{N}_0^d$. This gives the desired result directly. □

Let $z = \frac{j}{2}$ for $j = 2, 4, \ldots, \lfloor d \rfloor^e$, where $\lfloor x \rfloor^e$ is the largest even integer less than or equal to x. To find the residues of $\zeta_{\mathcal{E}}(s,a|\vec{r})$ at $s = z$, we compute the integral

$$\int_{\mathcal{C}}(-t)^{z-1}e^{-at}\sum_{\vec{m}\in\mathbb{N}_0^d} e^{-Q(\vec{m},\vec{r})t}dt.$$

Note that, with the use of (2.21), we get that

$$\int_{\mathcal{C}}(-t)^{z-1}e^{-at}\sum_{\vec{m}\in\mathbb{N}_0^d} e^{-Q(\vec{m},\vec{r})t}dt = \frac{(-1)^{z-1}}{2^d}\int_{\mathcal{C}} t^{z-1}\sum_{n\in\mathbb{N}_0}\frac{(-at)^n}{n!}\sum_{k=0}^{d} c_k t^{\frac{-k}{2}}dt.$$

Since the integrand is analytic everywhere but at $t = 0$, using the Cauchy Residue Theorem, we continue

$$\int_{\mathcal{C}} (-t)^{z-1} e^{-at} \sum_{\vec{m}\in\mathbb{N}_0^d} e^{-Q(\vec{m},\vec{r})t} dt = \frac{(-1)^{z-1}}{2^d} \int_{\mathcal{C}} t^{z-1} \sum_{n\in\mathbb{N}_0} \frac{(-at)^n}{n!} \sum_{k=0}^{d} c_k t^{\frac{-k}{2}} dt$$
$$= \frac{2\pi i(-1)^{z-1}}{2^d} \operatorname*{Res}_{t=0} \left\{ \sum_{n\in\mathbb{N}_0} \sum_{k=0}^{d} c_k \frac{(-a)^n}{n!} t^{n-\frac{k}{2}+z-1} \right\}.$$

Note that the only contributing term in the sum is $\frac{k}{2} - n = z$. In general there are several different options for k and n that will satisfy this relation. This consideration gets cumbersome as cases for even and odd d need to be considered separately. We will ultimately be interested only in the cases with $a = 0$, and so will consider this case only in more detail.

For $a = 0$, the calculation is simplified. The integral is

$$\int_{\mathcal{C}} (-t)^{z-1} \sum_{\vec{m}\in\mathbb{N}_0^d} e^{-Q(\vec{m},\vec{r})t} dt = \frac{(-1)^{z-1}}{2^d} \int_{\mathcal{C}} t^{z-1} \sum_{k=0}^{d} c_k t^{\frac{-k}{2}} dt$$
$$= \frac{2\pi i(-1)^{z-1}}{2^d} \operatorname*{Res}_{t=0} \left\{ \sum_{k=0}^{d} c_k t^{z-\frac{k}{2}-1} \right\},$$

so that $\frac{k}{2} = z$ $(2z = k)$ is the only term that contributes. Thus

$$\frac{1}{2^d} \int_{\mathcal{C}} (-t)^{z-1} \prod_{k=1}^{d} \left(\sqrt{\frac{\pi}{r_k t}} + 1 \right) dt = \frac{2\pi i(-1)^{z-1}}{2^d} c_{2z}.$$

This gives

$$\operatorname*{Res}_{s=z} \{\zeta_{\mathcal{E}}(s, a|\vec{r})\} \sim \frac{(-1)^z}{2^d} c_{2z} \operatorname*{Res}_{s=z} \{\Gamma(1-s)\} = \frac{c_{2z}}{2^d (z-1)!},$$

which is only valid for $z \in \mathbb{N}$ (i.e. $\Gamma(1 - z)$ is analytic everywhere else, see (2.7)), so that using this method yields only the residues at natural number values, missing the residues at $z = \frac{j}{2}$ where j is odd.

Due to the shortcomings of the above method for finding all of the residues of $\zeta_{\mathcal{E}}$, we proceed differently.

The residues of $\zeta_{\mathcal{E}}$ should occur at $z = \frac{j}{2}$, where $j = 1, 2, \ldots d$. Let us start with

$$\zeta_{\mathcal{E}}(s, a|\vec{r}) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-at} \sum_{\vec{m} \in \mathbb{N}_0} e^{-Q(\vec{m}, \vec{r})t} dt. \tag{2.24}$$

Note that with respect to the residues of $\zeta_{\mathcal{E}}$, only the small t behavior is needed so that we may break up (2.24) to

$$\zeta_{\mathcal{E}}(s, a|\vec{r}) = \frac{1}{\Gamma(s)} \left\{ \int_0^\varepsilon + \int_\varepsilon^\infty \right\} t^{s-1} e^{-at} \sum_{\vec{m} \in \mathbb{N}_0} e^{-Q(\vec{m}, \vec{r})t} dt$$

and consider only

$$E(s, a|\vec{r}) = \frac{1}{\Gamma(s)} \int_0^\varepsilon t^{s-1} e^{-at} \sum_{\vec{m} \in \mathbb{N}_0} e^{-Q(\vec{m}, \vec{r})t} dt, \tag{2.25}$$

realizing that the residues of $\zeta_{\mathcal{E}}$ are precisely those of E. Applying (2.21), we have

$$\begin{aligned} E(s, a|\vec{r}) &= \frac{1}{\Gamma(s)} \int_0^\varepsilon t^{s-1} e^{-at} \sum_{\vec{m} \in \mathbb{N}_0} e^{-Q(\vec{m}, \vec{r})t} dt \\ &= \frac{1}{2^d \Gamma(s)} \int_0^\varepsilon t^{s-1} \sum_{n \in \mathbb{N}_0} \frac{(-at)^n}{n!} \sum_{k=0}^d c_k t^{\frac{-k}{2}} dt \\ &= \frac{1}{2^d \Gamma(s)} \sum_{n \in \mathbb{N}_0} \sum_{k=0}^d c_k \frac{(-a)^n}{n!(n - \frac{k}{2} + s)} \varepsilon^{n - \frac{k}{2} + s}. \end{aligned}$$

Here we run into the same problem as above; that is, there are many different options for k and n that will satisfy this relation, so that the same complication arises.

Again for $a = 0$, the calculation is simplified, and we have, as a corollary to the above, that

$$E(s, 0|\vec{r}) = \frac{1}{2^d \Gamma(s)} \sum_{k=0}^d c_k \frac{\varepsilon^{s - \frac{k}{2}}}{(s - \frac{k}{2})}, \tag{2.26}$$

so that we have shown the following Proposition.

Proposition 2.11. *For* $s = \frac{j}{2}$,

$$\operatorname*{Res}_{s = \frac{j}{2}} \{\zeta_{\mathcal{E}}(s, 0|\vec{r})\} = \frac{1}{2^d \Gamma\left(\frac{j}{2}\right)} c_j,$$

where $j = 1, 2, \ldots, d$ *and* c_j *is as described in* (2.23).

In the same fashion we could have considered the Epstein functions resulting from the other boundary conditions.

In later sections we will be interested in the finite part of $\zeta_{\mathcal{E}}(s, 0|\vec{r})$. A formula for the finite part of the d-dimensional Epstein zeta-function would be notationally cumbersome, so in lieu of this we give a recursion formula, using Dirichlet boundary conditions, which may easily be applied to any particular dimension. Once down to the one-dimensional case the finite part of the Riemann zeta-function is applicable.

Since we will now be dealing with multiple Epstein zeta-functions of different dimension, we adopt the convention of superscripted dimension; that is, we denote the d-dimensional Epstein zeta-function by $\zeta_{\mathcal{E}}^{(d)}$.

We use the following definition of Kelvin functions from [10].

Definition 2.9. For $|\text{ph } z| < \frac{\pi}{2}$ and $\Re z^2 > 0$, denote

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^\nu \int_0^\infty t^{-\nu-1} e^{-t-\frac{z^2}{4t}}\, dt. \tag{2.27}$$

Proposition 2.12. *Let $r_d \neq 0$. Then, for Dirichlet boundary conditions,*

$$\begin{aligned}\zeta_{\mathcal{E}}^{(d)}(s, 0|\vec{r}) = \frac{\Gamma\left(s-\frac{1}{2}\right)}{2\Gamma(s)}\sqrt{\frac{\pi}{r_d}} \cdot \zeta_{\mathcal{E}}^{(d-1)}\left(s - \frac{1}{2}, 0|\vec{r}\right) + \\ -\frac{1}{2}\cdot \zeta_{\mathcal{E}}^{(d-1)}(s, 0|\vec{r}) + \frac{2}{\Gamma(s)}\pi^s r_d^{-\frac{1}{2}\left(s+\frac{1}{2}\right)} \times \\ \times \sum_{\vec{m}\in\mathbb{N}^d} \left(Q^{(d-1)}(\vec{m}, \vec{r})\right)^{\frac{1}{2}\left(\frac{1}{2}-s\right)} m_d^{s-\frac{1}{2}} K_{\frac{1}{2}-s}\left(2\pi m_d \sqrt{\frac{Q^{(d-1)}(\vec{m}, \vec{r})}{r_d}}\right).\end{aligned} \tag{2.28}$$

Proof. We start by writing

$$\begin{aligned}\zeta_{\mathcal{E}}^{(d)}(s, 0|\vec{r}) &= \sum_{\vec{m}\in\mathbb{N}^d} \frac{1}{(Q^{(d)}(\vec{m}, \vec{r}))^s} \\ &= \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1} \sum_{\vec{m}\in\mathbb{N}^d} e^{-Q^{(d)}(\vec{m}, \vec{r})t} dt.\end{aligned}$$

Applying (2.16) gives

$$\sum_{m\in\mathbb{N}} e^{-r_d m_d^2 t} = \sqrt{\frac{\pi}{r_d t}} \sum_{m\in\mathbb{N}} e^{-\frac{\pi^2 m_d^2}{r_d t}} + \frac{1}{2}\sqrt{\frac{\pi}{r_d t}} - \frac{1}{2},$$

which yields

$$\zeta_{\mathcal{E}}^{(d)}(s,0|\vec{r}) = \frac{1}{2\Gamma(s)}\sqrt{\frac{\pi}{r_d t}}\int_0^\infty t^{s-\frac{1}{2}-1}\sum_{\vec{m}\in\mathbb{N}^{d-1}} e^{-Q^{(d-1)}(\vec{m},\vec{r})t}dt+ \\ -\frac{1}{2\Gamma(s)}\int_0^\infty t^{s-1}\sum_{\vec{m}\in\mathbb{N}^{d-1}} e^{-Q^{(d-1)}(\vec{m},\vec{r})t}dt+ \\ +\frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{r_d t}}\int_0^\infty t^{s-\frac{1}{2}-1}\sum_{\vec{m}\in\mathbb{N}^{d}} e^{-Q^{(d-1)}(\vec{m},\vec{r})t}e^{-\frac{\pi^2 m_d^2}{r_d t}}dt.$$

where $Q^{(d-1)}(\vec{m},\vec{r}) = Q(\vec{m},\vec{r}) - r_d m_d^2$; that is, m_d is left out of the summation. Noting that the first two terms in our above sum are multiples of $(d-1)$-dimensional Epstein zeta-functions, and applying (2.27) to the third term, yields our desired result. $\square$

Note that for all $s \in \mathbb{C}$ the sums in (2.28) are rapidly convergent as the Kelvin functions are exponentially damped for large arguments.

Again, in a similar fashion, the other boundary conditions could have been dealt with.

2.5 General Zeta-function

In this section we examine a general zeta-function, determining residues and particular values. We first introduce the general type of sequence we want to use, giving it certain basic restrictions.

Let Λ be a monotonically increasing sequence of positive numbers such that the following hold.

(i) The partition function,

$$\Theta(t) = \sum_{\lambda\in\Lambda} e^{-\lambda t}, \tag{2.29}$$

converges for $\Re t > 0$.

(ii) For $t \to 0$, $\Theta(t)$ admits a full asymptotic expansion,

$$\Theta(t) \sim \sum_{n\in\mathbb{N}_0} A_{i_n} t^{i_n}, \tag{2.30}$$

where $\{i_n\} \subset \mathbb{R}$ and $\{i_n\}$ is monotonically increasing to infinity with $i_0 < 0$. For later convenience we denote $-i_0 = \mu_0$, $-i_1 = \mu_1$, etc. as needed.

We are now in a position to define a general zeta-function (of Λ-type) as the following.

Definition 2.10. Let Λ be a sequence as described above, and let $s \in \mathbb{C}$ with $\Re s > \mu_0$. We define the general Λ-type zeta-function as

$$\zeta_\Lambda(s) = \sum_{\lambda \in \Lambda} \frac{1}{\lambda^s}. \tag{2.31}$$

In order to find residues and particular values of ζ_Λ we will use the integral representation,

$$\zeta_\Lambda(s) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \Theta(t) dt, \tag{2.32}$$

gained by applying (2.5). Since we are only interested in small t-behavior for finding residues, and values of $\zeta_\Lambda(s)$ at $s = -n$, we may focus our examination on the function

$$L(s) = \frac{1}{\Gamma(s)} \int_0^\varepsilon t^{s-1} \Theta(t) dt, \tag{2.33}$$

realizing that the residues, and values of $\zeta_\Lambda(s)$ at $s = -n$, are precisely those of L (Note that this is the same approach we used with the Epstein zeta-function).

Substituting (2.30) into the above equation and performing the integration, we yield

$$L(s) = \frac{1}{\Gamma(s)} \int_0^\varepsilon t^{s-1} \sum_{n \in \mathbb{N}_0} A_{i_n} t^{i_n} dt = \frac{1}{\Gamma(s)} \sum_{n \in \mathbb{N}_0} A_{i_n} \frac{\varepsilon^{s+i_n}}{(s+i_n)} dt, \tag{2.34}$$

which gives the following propositions.

Proposition 2.13. *The residues of* ζ_Λ *occur at* $s = -i_n$*, and furthermore,*

$$\operatorname*{Res}_{s=-i_n} \{\zeta_\Lambda(s)\} = \frac{A_{i_n}}{\Gamma(-i_n)}. \tag{2.35}$$

Proposition 2.14. *For* $n \in \mathbb{N}_0$*,*

$$\zeta_\Lambda(-n) = (-1)^n n! A_n. \tag{2.36}$$

CHAPTER THREE

Riemann Type Moments

It is hoped that the following results are of interest in statistical mechanics as well as in number theory.

L.B. Richmond [22], 1975

3.1 Introduction

In this chapter we take under consideration the question, how many ways may one write a positive integer n as the sum of smaller integers? Within this consideration the Riemann zeta-function plays a very natural role, and so we begin to apply the background presented in Chapter 1 to what we call Riemann Type Moments. We denote these particular moments in this way to emphasize the use of the Riemann zeta-function and its various properties. Higher moments and variance are also considered.

3.2 General Results of all k

Definition 3.1. Let $p_m(n)$ denote the number of partitions of n into m parts. The k-th moment of $p_m(n)$ is denoted by $t_k(n)$, and is defined by

$$t_k(n) = \sum_{m \in \mathbb{N}_0} m^k p_m(n),$$

where $k \in \mathbb{N}_0$.

Note that for $m > n$, $p_m(n) = 0$, and also that $p_0(n) = 0$.

This definition does not seem to be a reasonable way to evaluate $t_k(n)$. The method of attack we shall use will be to come up with a generating function for $t_k(n)$ and then use properties of that generating function to yield a more reasonable expression for calculating $t_k(n)$, at least for large n.

Let us start by defining

$$G(x,z) = \prod_{r=1}^{\infty}(1-zx^r)^{-1} = \sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} p_m(n)x^n z^m, \tag{3.1}$$

and introducing the operator

$$\vartheta = z\frac{d}{dz}. \tag{3.2}$$

Using (3.1), we have

$$\vartheta^k G(x,z) = \sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} m^k p_m(n)x^n z^m. \tag{3.3}$$

To start our analysis we let $z = 1$ and use the convention $G(x,1) \equiv G(x)$, to see that

$$\vartheta^k G(x,1) \equiv \vartheta^k G(x) = \sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} m^k p_m(n)x^n = \sum_{n\in\mathbb{N}_0} t_k(n)x^n, \tag{3.4}$$

so that we have constructed a generating function for $t_k(n)$.

To construct a more usable form of $t_k(n)$ for evaluation, we use the Laurent Series coefficients of $\vartheta^k G(x)$ above as well as Cauchy's formula with $\varepsilon > 0$ suitably chosen, to yield

$$t_k(n) = \frac{1}{2\pi i}\int_{C(0,\varepsilon)} \vartheta^k G(x)x^{-(n+1)}dx, \tag{3.5}$$

which easily becomes

$$t_k(n) = \frac{1}{2\pi i}\int_{C(0,\varepsilon)} \frac{1}{x}e^{n(-\ln x + \frac{1}{n}\ln \vartheta^k G(x))}dx. \tag{3.6}$$

The substitution $x = e^{-a}$ gives

$$t_k(n) = \frac{1}{2\pi i}\int_{s.p.} e^{n(a+\frac{1}{n}\ln \vartheta^k G(e^{-a}))}da, \tag{3.7}$$

where $s.p.$ indicates a path that goes through the saddle point, at $a = \alpha$, of the integrand. A discussion of the saddle point follows Theorem 3.1.

Beginning with the calculation of the saddle point $a = \alpha$, we will make systematic use of the following identity.

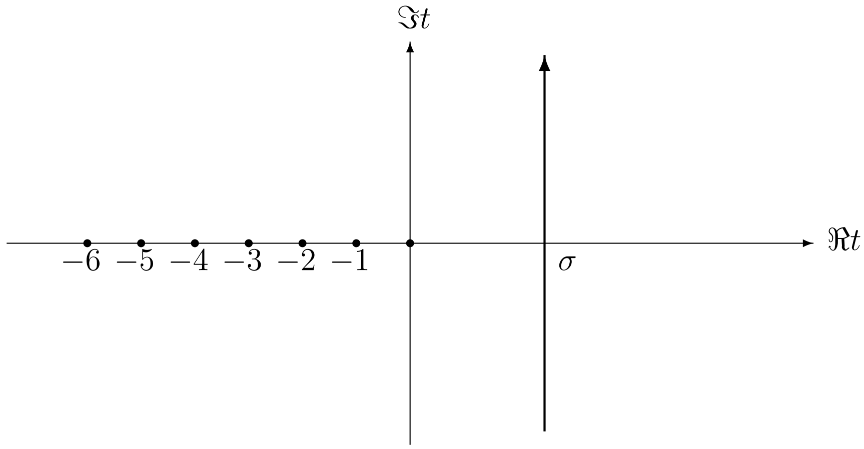

Figure 3.1. The Contour from $\sigma - i\infty$ to $\sigma + i\infty$ in the Complex Plane.

Proposition 3.1. *Let $\sigma > 0$, $\delta > 0$, and $|\text{ph } a| < \frac{\pi}{2} - \delta$. Then*

$$e^{-z} = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} z^{-t}\Gamma(t)dt \tag{3.8}$$

where the limits of integration define the contour shown in Figure 3.1.

Proof. Note that (2.7) gives $\operatorname*{Res}_{s=-n}\{\Gamma(s)\} = \frac{(-1)^n}{n!}$ for all $n \in \mathbb{N}_0$. Thus

$$\begin{aligned}\int_{\sigma - i\infty}^{\sigma + i\infty} z^{-t}\Gamma(t)dt &= \operatorname*{Res}_{t=\infty}\left\{z^{-t}\Gamma(t)\right\} + \sum_{n\in\mathbb{N}_0} z^n \operatorname*{Res}_{s=-n}\{\Gamma(s)\} \\ &= \sum_{n\in\mathbb{N}_0}\frac{(-1)^n}{n!}z^n = \operatorname*{Res}_{t=\infty}\left\{z^{-t}\Gamma(t)\right\} + e^{-z}.\end{aligned}$$

It is left to the reader to verify that $\operatorname*{Res}_{t=\infty}\left\{z^{-t}\Gamma(t)\right\} = 0$. □

The choice of α, in the contour *s.p.*, is dependent upon a saddle-point condition, i.e. in the case that $k = 0$[1] , α is the solution of

$$n = \sum_{r\in\mathbb{N}}\frac{r}{e^{\alpha r} - 1}. \tag{3.9}$$

Since we are interested in large n, we are interested in small α, hence we make an

[1] It is shown in Appendix A that n is independent of k up to $\mathcal{O}(\alpha)$, and so this solution works asymptotically for all k.

asymptotic expansion for α sufficiently small. Note that

$$n = \sum_{r\in\mathbb{N}} \frac{r}{e^{\alpha r}-1} = \sum_{r\in\mathbb{N}}\sum_{l\in\mathbb{N}} re^{-\alpha rl}$$
$$= \sum_{r\in\mathbb{N}}\sum_{l\in\mathbb{N}} \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-t} r^{-(t-1)} l^{-t}\Gamma(t)dt,$$

where the integral is taken from (3.8). This gives

$$n = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-t}\zeta_{\mathcal{R}}(t-1)\zeta_{\mathcal{R}}(t)\Gamma(t)dt, \tag{3.10}$$

where the contour is similar to Figure 3.1, but $\Re\sigma > 2$ so that we contain the residue at $t = 2$. The integrand has a simple pole at $t = 2$, and

$$\operatorname*{Res}_{t=2}\left\{\alpha^{-t}\zeta_{\mathcal{R}}(t-1)\zeta_{\mathcal{R}}(t)\Gamma(t)\right\} = \frac{\pi^2}{6\alpha^2},$$

Thus

$$n = \frac{\pi^2}{6\alpha^2}[1+\mathcal{O}(\alpha)], \tag{3.11}$$

giving the following lemma.

Lemma 3.1. *For sufficiently large n,*

$$\alpha = \frac{\pi}{(6n)^{\frac{1}{2}}}\left[1+\mathcal{O}(n^{-\frac{1}{2}})\right], \tag{3.12}$$

where α is a function of n given by (3.9).

Proof. Equation (3.12) follows directly from (3.11). □

In order to evaluate (3.7), we apply iteratively the following theorem of [21].

Theorem 3.1. *(Olver, [21] p.127) Suppose that*

(i) $p(t)$ and $q(t)$ are independent of z, and single valued and holomorphic in a domain $\mathbf{T}$.

(ii) The integration path $\mathcal{P}$ is independent of z. The endpoints a and b of $\mathcal{P}$ are finite or infinite, and $(a,b)_{\mathcal{P}}$ lies within $\mathbf{T}$.

(iii) $p'(t)$ *has a simple zero at an interior point* t_0 *of* $\mathcal{P}$.

(iv) z *ranges along a ray or over an annular sector given by* $\theta_1 \le \theta \le \theta_2$ *and* $|z| \ge Z$, *where* $\theta \equiv \text{ph } z$, $\theta_2 - \theta_1 < \pi$, *and* $Z > 0$. $I(Ze^{i\theta})$ *converges at* a *and* b *absolutely and uniformly with respect to* θ.

(v) $\Re\{e^{i\theta}p(t) - e^{i\theta}p(t_0)\}$ *is positive on* $(a,b)_{\mathcal{P}}$, *except at* t_0, *and is bounded away from zero uniformly with respect to* θ *as* $t \to a$ *or* b *along* $\mathcal{P}$.

Then

$$I(z) := \int_{\mathcal{P}} e^{-zp(t)} q(t) dt \sim 2e^{-zp(t_0)} \sum_{s=0}^{\infty} \Gamma\left(s + \frac{1}{2}\right) \frac{a_{2s}}{z^{s+(1/2)}} \tag{3.13}$$

as $z \to \infty$ *in the sector* $\theta_1 \le \text{ph } z \le \theta_2$.

We also note here that [21] gives the first two coefficients of the above sum as

$$a_0 = \frac{q}{\sqrt{2p''}}, \qquad a_2 = \left\{2q'' - \frac{2p'''q'}{p''} + \left(\frac{5p'''^2}{6p''^2} - \frac{p^{iv}}{2p''}\right) q\right\} \frac{1}{(2p'')^{3/2}}, \tag{3.14}$$

where p, q, and their derivatives are evaluated at t_0. [21] also gives the condition that in forming $\sqrt{2p''}$ and $(2p'')^{3/2}$, the branch of $\omega \equiv \text{ph }\{p''(t_0)\}$ must satisfy $|\omega_0 + \theta + 2\omega| \le \frac{\pi}{2}$, where ω is the limiting value of ph $(t - t_0)$ as $t \to t_0$ along $(t_0, b)_{\mathcal{P}}$.

We meet all, except *(i)*, of the assumptions of the above theorem. To circumnavigate (i), as previously stated, we apply the above Theorem iteratively, using power series expansions, to (3.7) to yield

$$t_k(n) = \frac{e^{n\alpha}\vartheta^k G(e^{-\alpha})}{2\pi} \cdot \frac{1}{i}\left[\sqrt{\frac{-2\pi}{nF''(\alpha)}} + \mathcal{O}(n^{-3/2})\right], \tag{3.15}$$

where $F(a) = a + \frac{1}{n}\ln \vartheta^k G(e^{-a})$, and $a = \alpha$ is the saddle point. We now have, with the saddle-point condition ($F'(\alpha) = 0$), that

$$F'(\alpha) = 1 + \frac{1}{n}\left(\frac{d}{da}\ln G(e^{-a})\right)|_{a=\alpha} = 0,$$

which gives $\frac{d}{da}\ln G(e^{-a}) = -n$, so that

$$\begin{aligned} F''(\alpha) &= -\frac{1}{n}\frac{d}{da}\left(-\frac{d}{da}\ln G(e^{-a})\right)|_{a=\alpha} \\ &= -\frac{1}{n}\frac{dn}{da}|_{a=\alpha}, \end{aligned}$$

thus

$$nF''(\alpha) = -\frac{dn}{da}|_{a=\alpha}.$$

Applying (3.12), we gain

$$\frac{dn}{da}|_{a=\alpha} = \frac{-\pi^2}{3\alpha^3}[1+\mathcal{O}(\alpha)],$$

and so

$$\left[\sqrt{\frac{-2\pi}{\frac{dn}{da}|_{a=\alpha}}} + \mathcal{O}\left(n^{-\frac{3}{2}}\right)\right] = \frac{\pi}{6^{\frac{1}{4}}}n^{-\frac{3}{4}} + \mathcal{O}\left(n^{-\frac{3}{2}}\right) = \frac{\pi}{6^{\frac{1}{4}}}n^{-\frac{3}{4}}\left[1+\mathcal{O}\left(n^{-\frac{3}{4}}\right)\right]. \quad (3.16)$$

Note that (3.12) gives

$$e^{n\alpha} = e^{\frac{\pi}{\sqrt{6}}n^{1/2}}\left[1+\mathcal{O}(n^{-1})\right], \quad (3.17)$$

so that (3.15) becomes

$$t_k(n) = \vartheta^k G(e^{-\alpha}) \cdot 2^{-1}6^{-\frac{1}{4}}n^{-\frac{3}{4}}e^{\frac{\pi}{\sqrt{6}}n^{1/2}}\left[1+\mathcal{O}\left(n^{-\frac{3}{4}}\right)\right]. \quad (3.18)$$

We now need only evaluate the above expression for each $k \in \mathbb{N}_0$. We break this up into three cases which we discuss in the sections that follow.

3.2.1 Case $k = 0$, (Hardy-Ramanujan)

To evaluate the 0th moment, $t_0(n)$, we apply (3.18), with $k = 0$. Note that (3.3) with $k = 0$ is merely (3.1).

We make the substitution $x = e^{-\alpha}$ with $z = 1$, in (3.1), so that

$$G(e^{-\alpha}, 1) \equiv G(e^{-\alpha}) = \prod_{r=1}^{\infty}(1-e^{-\alpha r})^{-1}. \quad (3.19)$$

For ease of use we will not look directly at $G(e^{-\alpha})$, but at $\ln G(e^{-\alpha})$. We then have

$$\ln G(e^{-\alpha}) = -\sum_{r=1}^{\infty}\ln(1-e^{-\alpha r}) = \sum_{r=1}^{\infty}\sum_{l=1}^{\infty}\frac{e^{-\alpha rl}}{l}. \quad (3.20)$$

The identity in (3.8) gives

$$\frac{e^{-\alpha rl}}{l} = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\alpha^{-t}r^{-t}l^{-(t+1)}\Gamma(t)dt. \quad (3.21)$$

Summing over $r \in \mathbb{N}$ and $l \in \mathbb{N}$ gives us

$$\ln G(e^{-\alpha}) = \sum_{r=1}^{\infty}\sum_{l=1}^{\infty} \frac{e^{-\alpha r l}}{l} = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-t}\zeta_{\mathcal{R}}(t)\zeta_{\mathcal{R}}(t+1)\Gamma(t)dt, \tag{3.22}$$

which is valid for $\sigma > 1$ and $|\mathrm{ph}\ a| < \frac{\pi}{2} - \delta$, for any $\delta > 0$.

Let us now focus on the integral in (3.22). Note that as we shift the contour lo the left we pick up the contribution from residues. For $0 < \Re\sigma < 1$ we have the residue at $t = 1$ contributing so that

$$\begin{aligned}\ln G(e^{-\alpha}) &= \frac{1}{\alpha}\zeta_{\mathcal{R}}(2)\Gamma(1) + \frac{1}{2\pi i}\left[\int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-t}\zeta_{\mathcal{R}}(t)\zeta_{\mathcal{R}}(t+1)\Gamma(t)dt\right] \\ &= \frac{\pi^2}{6\alpha} + \frac{1}{2\pi i}\left[\int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-t}\zeta_{\mathcal{R}}(t)\zeta_{\mathcal{R}}(t+1)\Gamma(t)dt\right].\end{aligned}$$

For $-1 < \Re\sigma < 0$ we have the pole of order 2 at $t = 0$ that contributes. Note that the residue at $t = 0$ is

$$\operatorname*{Res}_{t=0}\left\{\alpha^{-t}\zeta_{\mathcal{R}}(t)\zeta_{\mathcal{R}}(t+1)\Gamma(t)\right\} = -\frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln(\alpha),$$

so that

$$\ln G(e^{-\alpha}) = \frac{\pi^2}{6\alpha} - \frac{1}{2}\ln(2\pi) + \frac{1}{2}\ln(\alpha) + \mathcal{O}(\alpha).$$

Hence

$$G(e^{-\alpha}) = \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} e^{\frac{\pi^2}{6\alpha}}\left[1 + \mathcal{O}(\alpha)\right]. \tag{3.23}$$

Lemma 3.2. *For sufficiently large n,*

$$e^{n\alpha}G(e^{-\alpha}) = 2^{-\frac{1}{2}}6^{-\frac{1}{4}}n^{-\frac{1}{4}}e^{\frac{2\pi}{\sqrt{6}}n^{1/2}}\left[1 + \mathcal{O}(n^{-1/2})\right]. \tag{3.24}$$

Proof. Equation (3.24) follows directly from (3.23) and (3.12). □

Now putting together (3.18) with (3.24) and (3.16) yields the following result.

Theorem 3.2. *(Hardy-Ramanujan [14]) The asymptotic number of ways to partition n over $\mathbb{N}$ is*

$$t_0(n) = \frac{1}{4\sqrt{3}}n^{-1}e^{\pi\sqrt{\frac{2n}{3}}}\left[1 + \mathcal{O}(n^{-\frac{1}{2}})\right]. \tag{3.25}$$

Proof. Putting together (3.18) with (3.24) and (3.16) yields

$$\begin{aligned} t_0(n) &= \frac{e^{na}G(e^{-a})}{2\pi} \cdot \frac{1}{i}\left[\sqrt{\frac{2\pi}{\frac{dn}{da}|_{s.p.}}} + \mathcal{O}\left(n^{-\frac{3}{2}}\right)\right] \\ &= \frac{2^{-\frac{1}{2}}6^{-\frac{1}{4}}n^{-\frac{1}{4}}e^{\frac{2\pi}{\sqrt{6}}n^{1/2}}}{2\pi}\left[1+\mathcal{O}(n^{-1/2})\right]\frac{\pi}{6^{\frac{1}{4}}}n^{-\frac{3}{4}}\left[1+\mathcal{O}\left(n^{-\frac{3}{4}}\right)\right] \\ &= \frac{1}{4\sqrt{3}}n^{-1}e^{\pi\sqrt{\frac{2n}{3}}}\left[1+\mathcal{O}(n^{-\frac{1}{2}})\right], \end{aligned}$$

which is the desired result. □

3.2.2 Case $k = 1$

To evaluate the 1st moment, $t_1(n)$, we apply (3.18), with $k = 1$. Setting $k = 1$, note that (3.3) gives

$$\begin{aligned} \vartheta G(x, z) &= z\frac{d}{dz}\left\{\prod_{r=1}^{\infty}(1-zx^r)^{-1}\right\} \\ &= z\left(\sum_{r=1}^{\infty}\frac{x^r}{1-zx^r}\right)\left(\prod_{r=1}^{\infty}(1-zx^r)^{-1}\right) \\ &= G(x,z)\sum_{r=1}^{\infty}\frac{zx^r}{1-zx^r}. \end{aligned}$$

For convenience in later calculation we denote the proceeding sum $S(x, z)$; formally

$$S(x,z) = \sum_{r=1}^{\infty}\frac{zx^r}{1-zx^r}. \tag{3.26}$$

Note that from above we have

$$\vartheta G(x,1) \equiv \vartheta G(x) = G(x)\sum_{r\in\mathbb{N}}\frac{x^r}{1-x^r} = G(x)\sum_{r\in\mathbb{N}}\sum_{l\in\mathbb{N}}x^{-rl}. \tag{3.27}$$

The substitution $x = e^{-\alpha}$ with (3.8) gives

$$\begin{aligned} \vartheta G(e^{-\alpha}) &= G(e^{-\alpha})\sum_{r\in\mathbb{N}}\sum_{l\in\mathbb{N}}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\alpha^{-t}r^{-t}l^{-t}\Gamma(t)dt \\ &= G(e^{-\alpha})\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\alpha^{-t}\zeta_{\mathcal{R}}(t)\zeta_{\mathcal{R}}(t)\Gamma(t)dt. \end{aligned}$$

The above integrand has a pole of order 2 at $t = 1$. Note that

$$\operatorname*{Res}_{t=1}\left\{\alpha^{-t}\zeta_{\mathcal{R}}(t)\zeta_{\mathcal{R}}(t)\Gamma(t)\right\} = \frac{\gamma-\ln\alpha}{\alpha}, \tag{3.28}$$

where γ is Euler's constant, so that

$$e^{n\alpha}\vartheta G(e^{-\alpha}) = e^{n\alpha}G(e^{-\alpha})\left(\frac{\gamma - \ln\alpha}{\alpha}\right)[1+\mathcal{O}(\alpha)] \tag{3.29}$$

Now applying (3.12) we have

$$\left(\frac{\gamma - \ln\alpha}{\alpha}\right)[1+\mathcal{O}(\alpha)] = \frac{\sqrt{6n}}{\pi}\left(\gamma + \ln\frac{\sqrt{6n}}{\pi}\right)\left[1+\mathcal{O}(n^{-1/2})\right]. \tag{3.30}$$

Along with (3.24), the previous calculation for $e^{n\alpha}G(e^{-\alpha})$ in terms of n, gives

$$e^{n\alpha}\vartheta G(e^{-\alpha}) = \frac{2^{-\frac{1}{2}}6^{\frac{1}{4}}n^{\frac{1}{4}}}{\pi}e^{\pi\sqrt{\frac{2n}{3}}}\left(\gamma + \ln\frac{\sqrt{6n}}{\pi}\right)\left[1+\mathcal{O}(n^{-1/2})\right].$$

We need to simply put together the pieces, with $k = 1$ in (3.18), using the above and (3.16), to give the following result.

Theorem 3.3. *Let* $k = 1$, *then*

$$t_1(n) = \frac{\sqrt{2}}{4\pi n^{\frac{1}{2}}}e^{\pi\sqrt{\frac{2n}{3}}}\left(\gamma + \ln\frac{\sqrt{6n}}{\pi}\right)\left[1+\mathcal{O}(n^{-1/2})\right]. \tag{3.31}$$

3.2.3 Case $k \geq 2$

Having completed the cases for $k = 0, 1$ we now proceed to give a result for all $k \geq 2$. Note that computation of this result will involve k-th derivatives of $S(x, z)$ (equation (3.26)). To do this we use the following theorem.

Theorem 3.4. *For* $\vartheta = z\frac{d}{dz}$ *and* $G(x, z)$ *as in* (3.1),

$$\vartheta^k G(e^{-\alpha}) = G(e^{-\alpha})S^{(k)}(e^{-\alpha})$$

with

$$S^{(k)}(e^{-\alpha}) = \sum \frac{k!}{b_1!b_2!\cdots b_k!}\left(\frac{\vartheta^0 S(e^{-\alpha})}{1!}\right)^{b_1}\left(\frac{\vartheta^1 S(e^{-\alpha})}{2!}\right)^{b_2}\cdots\left(\frac{\vartheta^{k-1} S(e^{-\alpha})}{k!}\right)^{b_k}, \tag{3.32}$$

where the summation over all nonnegative integer solutions of $b_1 + 2b_2 + \cdots + kb_k = k$.

For a proof of the above theorem see Riordan [25].

We apply (3.32) to yield for $k \geq 2$,

$$\vartheta^k G(e^{-\alpha}) = G(e^{-\alpha}) S^{(k)}(e^{-\alpha})$$
$$= G(e^{-\alpha}) \sum \frac{k!}{b_1! b_2! \cdots b_k!} \left(\frac{\vartheta^0 S(e^{-\alpha})}{1!}\right)^{b_1} \left(\frac{\vartheta^1 S(e^{-\alpha})}{2!}\right)^{b_2} \cdots \left(\frac{\vartheta^{k-1} S(e^{-\alpha})}{k!}\right)^{b_k},$$

where the summation is over all solutions in $b_1, b_2, \ldots, b_m \in \mathbb{N}_0$ of $b_1 + 2b_2 + \cdots + kb_k = k$. For readability we denote the summation above $\sum(\vartheta^0 S(e^{-\alpha}), \vartheta^1 S(e^{-\alpha}), \ldots, \vartheta^{k-1} S(e^{-\alpha}))$. With this in mind we proceed as before.

Equation (3.32) gives

$$\vartheta^{(s-1)} S(x,1) = \sum_{r\in\mathbb{N}} \frac{\sum_{j=1}^{s} c_j^{(s)} x^{jr}}{(1-x^r)^s} = \sum_{j=1}^{s} c_j^{(s)} \sum_{r\in\mathbb{N}} \frac{x^{jr}}{(1-x^r)^s} = \sum_{j=1}^{s} c_j^{(s)} S_s^j(x), \tag{3.33}$$

where $S_s^j(x) = \sum_{r\in\mathbb{N}} \frac{x^{jr}}{(1-x^r)^s}$, and $c_j^{(s)}$'s are defined as in [22] by $c_0^{(s)} = 0$, $c_1^{(1)} = 1$, $c_1^{(2)} = 1$, $c_2^{(2)} = 0$, and for $s \geq 2$,

$$c_j^{(s+1)} = \begin{cases} j c_j^{(s)} + (s-j+1) c_{j-1}^{(s)}, & 1 \leq j \leq s \\ 0, & j = s+1. \end{cases} \tag{3.34}$$

Using the substitution $x = e^{-\alpha}$, we are interested in sums of the form

$$S_s^j(e^{-\alpha}) = \sum_{r\in\mathbb{N}} \frac{e^{-\alpha jr}}{(1-e^{-\alpha r})^s}. \tag{3.35}$$

Note that

$$\frac{1}{(1-x)^s} = \sum_{l\in\mathbb{N}_0} \binom{l+s-1}{s-1} x^l,$$

so that

$$S_s^j(e^{-\alpha}) = \sum_{r\in\mathbb{N}} \frac{e^{-\alpha jr}}{(1-e^{-\alpha r})^s} = \sum_{r\in\mathbb{N}} e^{-\alpha jr} \sum_{l\in\mathbb{N}_0} \binom{l+s-1}{s-1} e^{-\alpha rl}$$
$$= \sum_{r\in\mathbb{N}} \sum_{l\in\mathbb{N}_0} \binom{l+s-1}{s-1} e^{-\alpha r(l+j)}.$$

Applying (3.8), we have

$$S_s^j(e^{-\alpha}) = \sum_{r\in\mathbb{N}}\sum_{l\in\mathbb{N}_0}\begin{pmatrix} l+s-1 \\ s-1 \end{pmatrix}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\alpha^{-t}r^{-t}(l+j)^{-t}\Gamma(t)dt$$
$$= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\alpha^{-t}\left(\sum_{r\in\mathbb{N}}r^{-t}\right)\left(\sum_{l\in\mathbb{N}_0}\begin{pmatrix} l+s-1 \\ s-1 \end{pmatrix}(l+j)^{-t}\right)\Gamma(t)dt.$$

Recalling Proposition 2.7 we have

$$\zeta_{\mathcal{B}}(t,j|\vec{1}) \equiv \zeta_{\mathcal{B}}(t,j) = \sum_{l\in\mathbb{N}_0}\begin{pmatrix} l+s-1 \\ s-1 \end{pmatrix}(l+j)^{-t},$$

where $\zeta_{\mathcal{B}}(t,j)$ is the zeta-function of Barnes of dimension s. This yields

$$S_s^j(e^{-\alpha}) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\alpha^{-t}\zeta_{\mathcal{R}}(t)\zeta_{\mathcal{B}}(t,j)\Gamma(t)dt, \tag{3.36}$$

which easily gives, for $s \geq 2$,

$$S_s^j(e^{-\alpha}) = \frac{\zeta_{\mathcal{R}}(s)\Gamma(s)}{\alpha^s}\operatorname*{Res}_{t=s}\{\zeta_{\mathcal{B}}(t,j)\} + \mathcal{O}\left(\alpha^{-s+1}\right)$$
$$= \frac{\zeta_{\mathcal{R}}(s)\Gamma(s)}{(s-1)!\alpha^s}B_0^{(s)}(j) + \mathcal{O}\left(\alpha^{-s+1}\right),$$

where $B_0^{(s)}(a|\vec{r}) = 1$ is the 0th s-dimensional Bernoulli polynomial evaluated at a and $\vec{r}$. Thus

$$S_s^j(e^{-\alpha}) = \alpha^{-s}\zeta_{\mathcal{R}}(s)\left[1+\mathcal{O}(\alpha)\right]. \tag{3.37}$$

Here we also note that for $s \geq 2$,

$$\sum_{j=1}^{s+1}c_j^{(s)} = \sum_{j=1}^{s+1}\left[jc_j^{(s)} + (s-j+1)c_{j-1}^{(s)}\right] \tag{3.38}$$
$$= \sum_{j=1}^{s}jc_j^{(s)} + \sum_{j=1}^{s}(s-j)c_j^{(s)} = s\sum_{j=1}^{s}c_j^{(s)} = s!. \tag{3.39}$$

Using the above,

$$\vartheta^s S(e^{-\alpha}) = \sum_{j=1}^{s}c_j^{(s)}S_s^j(x)$$
$$= \left(\sum_{j=1}^{s}c_j^{(s)}\right)\alpha^{-s}\zeta_{\mathcal{R}}(s)\left[1+\mathcal{O}(\alpha)\right]$$
$$= s!\alpha^{-s}\zeta_{\mathcal{R}}(s+1)\left[1+\mathcal{O}(\alpha)\right].$$

Applying the proceeding equation and (3.12) yields

$$\sum(\vartheta^0 S(e^{-\alpha}), \vartheta^1 S(e^{-\alpha}), \ldots, \vartheta^{k-1} S(e^{-\alpha}))$$
$$= \left(\frac{\sqrt{6n}}{\pi}\right)^k \sum\left(\gamma + \ln\frac{\sqrt{6n}}{\pi}, \zeta_{\mathcal{R}}(2), \ldots, (k-1)!\zeta_{\mathcal{R}}(k)\right) \times$$
$$\times \left[1 + \mathcal{O}(n^{-\frac{1}{2}})\right]. \quad (3.40)$$

The above result along with (3.24) and (3.16) applied to (3.18) gives the following result, correcting Theorem 2.1 of Richmond [22].

Theorem 3.5. *For $k \geq 2$,*

$$t_k(n) = \frac{1}{4\sqrt{3}} n^{-1} e^{\pi\sqrt{\frac{2n}{3}}} \left(\frac{\sqrt{6n}}{\pi}\right)^k \times$$
$$\times \sum\left(\gamma + \ln\frac{\sqrt{6n}}{\pi}, \zeta_{\mathcal{R}}(2), \ldots, (k-1)!\zeta_{\mathcal{R}}(k)\right) \cdot \left[1 + \mathcal{O}(n^{-\frac{1}{2}})\right].$$

3.3 Properties of Summands

In this section we give expressions for the expected number of summands as well as the variance of of the average number of summands.

Definition 3.2. The expected number of summands of a Riemann type partition of an integer n, denoted by $m(n)$, is

$$m(n) = \frac{t_1(n)}{t_0(n)}.$$

This leads directly to the following proposition.

Proposition 3.2. *The expected number of summands of a Riemann type partition of an integer n is*

$$m(n) = \frac{\sqrt{6n}}{\pi}\left(\gamma + \ln\frac{\sqrt{6n}}{\pi}\right)\left[1 + \mathcal{O}\left(n^{\frac{-1}{2}}\right)\right].$$

Proof. The solution follows from the direct division of $t_1(n)$ by $t_0(n)$. □

All of the needed details for the calculation of the variance of the average number of summands for a Riemann type partitioning are given in the discussion proceeding the calculation of the variance of a general sequence. So as not to do two identical calculations, we will merely define, and then state the variance for Riemann type sequences as a corollary to Theorem 4.4 of the next chapter.

Definition 3.3. The variance of the expected number of summands of a Riemann type partition of an integer n, denoted by $\sigma^2(n)$, is

$$\sigma^2(n) = \frac{t_2(n)}{t_0(n)} - \left(\frac{t_1(n)}{t_0(n)}\right)^2.$$

Corollary 3.1. *For a Riemann type partitioning,*

$$\sigma^2(n) = n\left[1 + \mathcal{O}\left(n^{\frac{-1}{2}}\right)\right].$$

CHAPTER FOUR

General Λ-Type Moments

There is, essentially, only one problem in statistical thermodynamics: the distribution of a given amount of energy E over N identical systems.

Erwin Schrödinger [26], 1946

4.1 Introduction

This chapter presents the main results of this paper. It answers the question, how many ways are there to write an integer n as the sum of members from a given sequence of nondecreasing natural numbers? We will denote this type of sequence by Λ, and will proceed using the results of the background section on general zeta-functions. After the consideration of the 0-th moment, higher moments and variance are calculated.

4.2 General Results for all k

Throughout this chapter we denote by Λ a sequence as defined in Section 2.5 with the additional stipulation that $1 \in \Lambda$.

Definition 4.1. Let $p_\Lambda(n,m)$ be the number of partitions of n into m summands where each summand is a member of Λ. The k-th moment of $p_\Lambda(n,m)$ is denoted by $t_\Lambda^k(n)$, and is defined by

$$t_\Lambda^k(n) = \sum_{m \in \mathbb{N}_0} m^k p_\Lambda(n,m),$$

where $p_\Lambda(n,m) = 0$ for $m > n$ and $p_\Lambda(n,0) = 0$.

Using this definition we run into the same problem as in the Riemannian chapter; that is, we need a more reasonable way to compute $t_\Lambda^k(n)$. We resolve this problem with the same approach, first we construct a generating function and then we continue by evaluating the coefficients of this generating function. Many of the calculations presented in this chapter are analogous to those of the previous chapter, so that for fear of being too repetitive, similar calculations are omitted.

As before we define

$$G_\Lambda(x,z) = \sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} p_\Lambda(n,m)x^n z^m = \prod_{\lambda\in\Lambda}(1-zx^\lambda)^{-1}, \tag{4.1}$$

and use the operator ϑ as in (3.2). Note that

$$\vartheta^k G_\Lambda(x,1) \equiv \vartheta^k G_\Lambda(x) = \sum_{n\in\mathbb{N}_0}\sum_{m\in\mathbb{N}_0} m^k p_\Lambda(n,m)x^n = \sum_{n\in\mathbb{N}_0} t_\Lambda^k(n)x^n,$$

so that we have constructed a generating function for $t_\Lambda^k(n)$.

Since Λ contains only integers, we may apply Cauchy's formula for Laurent Series coefficients, so that for $\varepsilon > 0$ suitably chosen, we have

$$t_\Lambda^k(n) = \frac{1}{2\pi i}\int_{C(0,\varepsilon)} \vartheta^k G_\Lambda(x)x^{-(n+1)}dx, \tag{4.2}$$

which, with the substitution $x = e^{-a}$, easily becomes

$$t_\Lambda^k(n) = \frac{1}{2\pi i}\int_{s.p.} e^{n(-\ln e^{-a}+\frac{1}{n}\ln\vartheta^k G_\Lambda(e^{-a}))}da, \tag{4.3}$$

where, as before, $s.p.$ indicates a path that goes through the saddle point, at $a = \alpha$, of the integrand.

Again the above choice of α is dependent upon a saddle-point condition; that is, α is the solution[1] of

$$n = \sum_{\lambda\in\Lambda}\frac{\lambda}{e^{\alpha\lambda}-1}. \tag{4.4}$$

Since we are interested in large n, we are interested in small α, hence we make an asymptotic expansion for α sufficiently small. Using (3.8)

$$n = \sum_{\lambda\in\Lambda}\sum_{l\in\mathbb{N}}\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\alpha^{-t}\lambda^{-(t-1)}l^{-t}\Gamma(t)dt,$$

[1] See Appendix A.

which gives

$$n = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-t} \zeta_\Lambda(t-1) \zeta_{\mathcal{R}}(t) \Gamma(t) dt, \tag{4.5}$$

where the contour is similar to Figure 3.1, but $\Re\sigma > \mu_0 + 1$ so that we contain all of the residues of the integrand. The integrand has a simple poles at $t = \mu_0 + 1, \mu_1 + 1, \ldots, 1, 0$, so that

$$n = \sum_{i=0}^{d} \frac{\zeta_{\mathcal{R}}(\mu_i + 1)}{\alpha^{\mu_i + 1}} \mu_i A_{-\mu_i} + \frac{A_0}{\alpha} + \mathcal{O}(\alpha^0), \tag{4.6}$$

where A_k's are given by (2.30). A general expression for α in terms of n may not be obtained, but for any given set of singularities (the μ_i's) the solution is easily realized, examples of which will be given in the next chapter. Thus, throughout this chapter α will be defined as the solution of (4.6).

Applying Theorem 3.1 as in Section 3.2, gives

$$t_\Lambda^k(n) = \frac{e^{n\alpha} \vartheta^k G_\Lambda(e^{-\alpha})}{2\pi} \cdot \left[\sqrt{\frac{-2\pi}{\frac{dn}{da}|_{a=\alpha}}} + \mathcal{O}(n^{-3/2}) \right]. \tag{4.7}$$

We now begin to evaluate some of the pieces of the above equation.

First, we simplify $\vartheta^k G_\Lambda(e^{-\alpha})$ as before, making two cases; $k = 1$ and $k \geq 2$. For $k = 1$,

$$\vartheta G_\Lambda(x, z) = z \frac{d}{dz} \left\{ \prod_{\lambda \in \Lambda} (1 - z x^\lambda)^{-1} \right\} \tag{4.8}$$

$$= z \left(\sum_{\lambda \in \Lambda} \frac{x^\lambda}{1 - z x^\lambda} \right) \left(\prod_{\lambda \in \Lambda} (1 - z x^\lambda)^{-1} \right) \tag{4.9}$$

$$= G_\Lambda(x, z) \sum_{\lambda \in \Lambda} \frac{z x^\lambda}{1 - z x^\lambda}. \tag{4.10}$$

For convenience in later calculation we denote the proceeding sum $S_\Lambda(x, z)$; formally

$$S_\Lambda(x, z) = \sum_{\lambda \in \Lambda} \frac{z x^\lambda}{1 - z x^\lambda}. \tag{4.11}$$

From above we have $\vartheta G_\Lambda(e^{-\alpha}) = G_\Lambda(e^{-\alpha}) S_\Lambda(e^{-\alpha})$.

For $k \geq 2$ we gain a similar result; that is, repeating the above process yields

$$\vartheta^k G_\Lambda(e^{-\alpha}) = G_\Lambda(e^{-\alpha}) S_\Lambda^{(k)}(e^{-\alpha}).$$

Reiterating that $\vartheta^k G_\Lambda(e^{-\alpha}) = G_\Lambda(e^{-\alpha}) S_\Lambda^{(k)}(e^{-\alpha})$ for all $k \geq 1$, we see that for $k \geq 1$ (4.7) becomes

$$t_\Lambda^k(n) = S_\Lambda^{(k)}(e^{-\alpha}) \cdot \frac{e^{n\alpha} G_\Lambda(e^{-\alpha})}{2\pi} \left[\sqrt{\frac{-2\pi}{\frac{dn}{da}|_{a=\alpha}}} + \mathcal{O}(n^{-3/2}) \right]. \tag{4.12}$$

It is apparent in the proceeding equation that for each moment, the quantity $e^{n\alpha} G_\Lambda(e^{-\alpha})$ must be known. For this reason, let us take a moment[2] to determine this quantity. Analogous to the calculation in the Riemannian chapter, we see that

$$\begin{aligned} \ln G_\Lambda(e^{-\alpha}) &= \sum_{\lambda \in \Lambda} \sum_{l \in \mathbb{N}} \frac{e^{-\alpha \lambda l}}{l} \\ &= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-t} \zeta_\Lambda(t) \zeta_\mathcal{R}(t+1) \Gamma(t) dt. \end{aligned}$$

The integrand has simple poles at $t = \mu_i$ for $i = 0, 1, \ldots, d$, and a double pole at $t = 0$. Calculation of the integral gives

$$\ln G_\Lambda(e^{-\alpha}) = \sum_{i=0}^{d} \frac{\zeta_\mathcal{R}(\mu_i + 1)}{\alpha^{\mu_i}} A_{-\mu_i} + A_0 \ln \frac{1}{\alpha} + \zeta'_\Lambda(0) + \mathcal{O}(\alpha). \tag{4.13}$$

Since $e^{n\alpha} G_\Lambda(e^{-\alpha}) = e^{n\alpha + \ln G_\Lambda(e^{-\alpha})}$, we have the following lemma.

Lemma 4.1. *For α the solution of* (4.6),

$$e^{n\alpha} G_\Lambda(e^{-\alpha}) = \alpha^{-A_0} \exp\left[\sum_{i=0}^{d} \frac{\mu_i + 1}{\alpha^{\mu_i}} \zeta_\mathcal{R}(\mu_i + 1) A_{-\mu_i} + A_0 + \zeta'_\Lambda(0) \right] [1 + \mathcal{O}(\alpha)]. \tag{4.14}$$

Changing focus to $\frac{dn}{da}|_{a=\alpha}$, we note that since this term is not exponentiated in the solution of $t_\Lambda^k(n)$, thus to obtain asymptotic results, we need only determine the leading order. With this in mind, (4.6) gives

$$-\frac{dn}{da}|_{a=\alpha} = \frac{\mu_0(\mu_0 + 1)}{\alpha^{\mu_0 + 2}} \zeta_\mathcal{R}(\mu_0 + 1) A_{-\mu_0} \left[1 + \mathcal{O}\left(\alpha^{\mu_0 - \mu_1} \right) \right], \tag{4.15}$$

so that

$$\frac{1}{2\pi} \left[\sqrt{\frac{-2\pi}{\frac{dn}{da}|_{a=\alpha}}} + \mathcal{O}(n^{-3/2}) \right] = \left(2\pi \mu_0(\mu_0 + 1) \zeta_\mathcal{R}(\mu_0 + 1) A_{-\mu_0}\right)^{-\frac{1}{2}} \alpha^{\frac{\mu_0}{2} + 1} \times$$
$$\times \left[1 + \mathcal{O}\left(\alpha^{\frac{4\mu_0 + 5}{2}} \right) \right]. \tag{4.16}$$

[2] No pun intended.

With the main calculations behind us, we now begin to consider specific k values. We break this up into two cases; $k = 0$ and $k \geq 1$. The proceeding calculation brings us to our first general moment theorem.

4.2.1 General Λ-type: $k = 0$

To evaluate the 0th moment of the general Λ-type, $t_\Lambda^0(n)$, we apply (4.12), with $k = 0$, along with Lemma 4.1 and (4.16) to yield the following theorem.

Theorem 4.1. *For Λ-type partitions,*

$$t_\Lambda^0(n) = (2\pi\mu_0(\mu_0+1)\zeta_{\mathcal{R}}(\mu_0+1)A_{-\mu_0})^{-\frac{1}{2}}\,\alpha^{\frac{\mu_0}{2}+1-A_0}\times$$
$$\times\exp\left[\sum_{i=0}^{d}\frac{\mu_i+1}{\alpha^{\mu_i}}\zeta_{\mathcal{R}}(\mu_i+1)A_{-\mu_i}+A_0+\zeta'_\Lambda(0)\right][1+\mathcal{O}(\alpha)].$$

4.2.2 General Λ-type: Case $k \geq 1$

In evaluating the k-th moment of the General Λ-type for $k \geq 1$, $t_\Lambda^k(n)$, we note that (4.12) gives the following proposition immediately.

Proposition 4.1. *For all k,*

$$t_\Lambda^k(n) = t_\Lambda^0(n)\cdot S_\Lambda^{(k)}(e^{-\alpha}). \tag{4.17}$$

As a consequence, to determine $t_\Lambda^k(n)$ we need only evaluate $S_\Lambda^{(k)}(e^{-\alpha})$ and then apply the above proposition. Note that as before

$$S_\Lambda^{(k)}(e^{-\alpha}) = \sum\frac{k!}{b_1!b_2!\cdots b_k!}\left(\frac{\vartheta^0 S_\Lambda(e^{-\alpha})}{1!}\right)^{b_1}\left(\frac{\vartheta^1 S_\Lambda(e^{-\alpha})}{2!}\right)^{b_2}\cdots\left(\frac{\vartheta^{k-1} S_\Lambda(e^{-\alpha})}{k!}\right)^{b_k}, \tag{4.18}$$

where the summation is over all solutions in $b_1, b_2, \ldots, b_m \in \mathbb{N}_0$ of $b_1 + 2b_2 + \cdots + kb_k = k$. Again for readability, we denote the summation above $\sum(\vartheta^0 S_\Lambda(e^{-\alpha}), \vartheta^1 S_\Lambda(e^{-\alpha}), \ldots, \vartheta^{k-1} S_\Lambda(e^{-\alpha}))$.

It is evident from the above equation that within the calculation for general k, we will need the specific calculation for $s = 1$; that is, we must evaluate $\vartheta^0 S_\Lambda(e^{-\alpha}) = S_\Lambda(e^{-\alpha})$.

Note that (3.8) gives

$$S_\Lambda(e^{-\alpha}) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-t} \zeta_\Lambda(t) \zeta_{\mathcal{R}}(t) \Gamma(t) dt.$$

To evaluate $S_\Lambda(e^{-\alpha})$ we must distinguish between a few cases. The above integral has different values depending on whether $\mu_0 < 1$, $\mu_0 = 1$, or $\mu_0 > 1$. We treat these cases independently.

For $\mu_0 < 1$, the leading pole of $\zeta_{\mathcal{R}}(t)$ at $t = 1$ is simple, and so

$$S_\Lambda(e^{-\alpha}) = \frac{\zeta_\Lambda(1)}{\alpha} \left[1 + \mathcal{O}\left(\alpha^{1-\mu_0}\right)\right]. \tag{4.19}$$

For $\mu_0 = 1$, the integrand has a double pole at $t = 1$, and

$$\begin{aligned} S_\Lambda(e^{-\alpha}) &= \frac{1}{\alpha} \left(\operatorname*{FP}_{t=1} \{\zeta_\Lambda(t)\} + \gamma A_{-1} + A_{-1}(\psi(1) - \ln \alpha) \right) [1 + \mathcal{O}(\alpha^{1-\mu_1})] && (4.20) \\ &= \frac{1}{\alpha} \left(\operatorname*{FP}_{t=1} \{\zeta_\Lambda(t)\} - A_{-1} \ln \alpha \right) [1 + \mathcal{O}(\alpha^{1-\mu_1})], && (4.21) \end{aligned}$$

where γ is Euler's number.

For $\mu_0 > 1$, the integrand has a simple pole, at $t = \mu_0$, and

$$\begin{aligned} S_\Lambda(e^{-\alpha}) &= \frac{\zeta_{\mathcal{R}}(\mu_0)\Gamma(\mu_0)}{\alpha^{\mu_0}} \operatorname*{Res}_{t=\mu_0} \{\zeta_\Lambda(t)\} \left[1 + \mathcal{O}\left(\alpha^{\min(\mu_0 - 1, \mu_0 - \mu_1)}\right)\right] && (4.22) \\ &= \frac{\zeta_{\mathcal{R}}(\mu_0) A_{-\mu_0}}{\alpha^{\mu_0}} \left[1 + \mathcal{O}\left(\alpha^{\min(\mu_0 - 1, \mu_0 - \mu_1)}\right)\right]. && (4.23) \end{aligned}$$

Having completed the evaluations for $s = 1$ we now focus on the terms $\vartheta^{s-1} S_\Lambda(e^{-\alpha})$ for $s > 1$. Note that (3.32) gives

$$\vartheta^{s-1} S_\Lambda(x, 1) = \sum_{\lambda \in \Lambda} \frac{\sum_{j=1}^{s} c_j^{(s)} x^{j\lambda}}{(1 - x^\lambda)^s} = \sum_{j=1}^{s} c_j^{(s)} \sum_{\lambda \in \Lambda} \frac{x^{j\lambda}}{(1 - x^\lambda)^s} = \sum_{j=1}^{s} c_j^{(s)} S_\Lambda^{s,j}(x), \tag{4.24}$$

where $S_\Lambda^{s,j}(x) = \sum_{\lambda \in \Lambda} \frac{x^{j\lambda}}{(1 - x^\lambda)^s}$, and $c_j^{(s)}$'s are as in (3.34). Using the substitution $x = e^{-\alpha}$, we are, as before, interested in sums of the form

$$S_\Lambda^{s,j}(e^{-\alpha}) = \sum_{\lambda \in \Lambda} \frac{e^{-\alpha j \lambda}}{(1 - e^{-\alpha\lambda})^s}. \tag{4.25}$$

Similar to the previous chapter

$$S_\Lambda^{s,j}(e^{-\alpha}) = \sum_{\lambda\in\Lambda} \frac{e^{-\alpha j\lambda}}{(1-e^{-\alpha\lambda})^s} = \sum_{\lambda\in\Lambda} e^{-\alpha j\lambda} \sum_{l\in\mathbb{N}_0} e_l^{(s)} e^{-\alpha\lambda l} = \sum_{\lambda\in\Lambda}\sum_{l\in\mathbb{N}_0} e_l^{(s)} e^{-\alpha\lambda(l+j)},$$

so that applying (3.8), we have

$$\begin{aligned} S_\Lambda^{s,j}(e^{-\alpha}) &= \sum_{\lambda\in\Lambda}\sum_{l\in\mathbb{N}_0} e_l^{(s)} \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-t}\lambda^{-t}(l+j)^{-t}\Gamma(t)dt \\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-t}\left(\sum_{\lambda\in\Lambda}\lambda^{-t}\right)\left(\sum_{l\in\mathbb{N}_0} e_l^{(s)}(l+j)^{-t}\right)\Gamma(t)dt. \end{aligned}$$

Since we arrive here at a place where we are dealing with two zeta-functions of different dimension, we adopt the following notation. We denote the dimension as a parenthetical superscript (e.g. $\zeta_\mathcal{B}^{(d)}$ is the zeta-function of Barnes of dimension d). With this new notation we yield

$$S_\Lambda^{s,j}(e^{-\alpha}) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \alpha^{-t}\zeta_\Lambda(t)\zeta_\mathcal{B}^{(s)}(t,j)\Gamma(t)dt. \tag{4.26}$$

Note that we have k values for s, $s = 1, 2, \ldots, k$. Previously in this section we have considered $s = 1$, now we turn our attention to the following three cases for $s \geq 2$: $s < \mu_0$, $s = \mu_0$, $s > \mu_0$.

For $s < \mu_0$, the leading term comes from the simple pole of $\zeta_\Lambda(t)$ at $t = \mu_0$, so that

$$\begin{aligned} S_\Lambda^{s,j}(e^{-\alpha}) &= \frac{\zeta_\mathcal{B}^{(s)}(\mu_0,j)\Gamma(\mu_0)}{\alpha^{\mu_0}}\operatorname*{Res}_{t=\mu_0}\{\zeta_\Lambda(t)\} + \mathcal{O}\left(\alpha^{\min(-\mu_1,-s)}\right) \\ &= \frac{\zeta_\mathcal{B}^{(s)}(\mu_0,j)A_{-\mu_0}}{\alpha^{\mu_0}}\left[1+\mathcal{O}\left(\alpha^{\min(\mu_0-\mu_1,\mu_0-s)}\right)\right]. \end{aligned}$$

Now we have, using the above and (4.24), that

$$\vartheta^{s-1}S_\Lambda(e^{-\alpha}) = \alpha^{-\mu_0}A_{-\mu_0}\sum_{j=1}^{s} c_j^{(s)}\zeta_\mathcal{B}^{(s)}(\mu_0,j)\left[1+\mathcal{O}\left(\alpha^{\min(\mu_0-\mu_1,\mu_0-s)}\right)\right]. \tag{4.27}$$

For $s = \mu_0$, the leading term comes from the double pole of $\zeta_\Lambda(t)\zeta_\mathcal{B}^{(s)}(t,j)$ at $t = \mu_0$, and so

$$\begin{aligned} S_\Lambda^{s,j}(e^{-\alpha}) = \frac{1}{\alpha^s}\left(\operatorname*{FP}_{t=s}\{\zeta_\Lambda(t)\} + A_{-s}\operatorname*{FP}_{t=s}\left\{\zeta_\mathcal{B}^{(s)}(t,j)\right\} + \frac{A_{-s}}{\Gamma(s)}(\psi(s)-\ln\alpha)\right)\times \\ \times\left[1+\mathcal{O}\left(\alpha^{\min(\mu_0-\mu_1,1)}\right)\right]. \end{aligned}$$

Thus

$$\vartheta^{s-1} S_\Lambda(e^{-\alpha}) = \frac{1}{\alpha^s}\Bigg((s-1)!\underset{t=s}{\mathrm{FP}}\left\{\zeta_\Lambda(t)\right\} + A_{-s}\left(\psi(s) - \ln\alpha\right) + \\ + \sum_{j=1}^{s} c_j^{(s)} \underset{t=s}{\mathrm{FP}}\left\{\zeta_{\mathcal{B}}^{(s)}(t,j)\right\}\Bigg)\left[1 + \mathcal{O}\left(\alpha^{\min(\mu_0-\mu_1,1)}\right)\right]. \quad (4.28)$$

Finally, for $s > \mu_0$, the leading term comes from the simple pole of $\zeta_{\mathcal{B}}^{(s)}(t,j)$ at $t = s$, so that

$$S_\Lambda^{s,j}(e^{-\alpha}) = \alpha^{-s}\zeta_\Lambda(s)\left[1 + \mathcal{O}\left(\alpha^{\min(s-\mu_0,1)}\right)\right],$$

and so

$$\vartheta^{s-1} S_\Lambda(e^{-\alpha}) = (s-1)!\alpha^{-s}\zeta_\Lambda(s)\left[1 + \mathcal{O}\left(\alpha^{\min(s-\mu_0,1)}\right)\right]. \quad (4.29)$$

We are now in a position to evaluate $S_\Lambda^{(k)}(e^{-\alpha})$. Again we consider three cases: $\mu_0 > 1$, $\mu_0 = 1$, $\mu_0 < 1$.

If $\mu_0 > 1$, then the leading term of $S_\Lambda^{(k)}(e^{-\alpha})$, as defined in (4.18), comes from $b_1 = k$. This is somewhat counterintuitive, but realizing that when $b_1 = k$, $b_i = 0$ for all $i = 2, 3, \ldots, k$, shows that the relationship between μ_0 and $s \in \{1, 2, \ldots, k\}$, is overshadowed by the fact that $\mu_0 > 1$. That said, we see that this implies that for $\mu_0 > 1$,

$$S_\Lambda^{(k)}(e^{-\alpha}) \sim \left(S_\Lambda(e^{-\alpha})\right)^k,$$

so that, (4.23) gives

$$S_\Lambda^{(k)}(e^{-\alpha}) \sim \alpha^{-\mu_0 k} A_{-\mu_0}^k \left[\zeta_{\mathcal{R}}(\mu_0)\right]^k. \quad (4.30)$$

If $\mu_0 = 1$, we have directly from (4.21) and (4.29) that

$$S_\Lambda^{(k)}(e^{-\alpha}) = \alpha^{-k}\sum\left(\underset{t=1}{\mathrm{FP}}\left\{\zeta_\Lambda(t)\right\} - A_{-1}\ln\alpha, \zeta_\Lambda(2), 2\zeta_\Lambda(3), \ldots, (k-1)!\zeta_\lambda(k)\right). \quad (4.31)$$

Finally, if $\mu_0 < 1$, (4.19) and (4.29) give

$$S_\Lambda^{(k)}(e^{-\alpha}) = \alpha^{-k}\sum\left(\zeta_\Lambda(1), \zeta_\Lambda(2), 2\zeta_\Lambda(3), \ldots, (k-1)!\zeta_\lambda(k)\right). \quad (4.32)$$

We may now apply Proposition 4.1 to give the following theorem.

Theorem 4.2. *For $k \geq 1$ and α the solution of* (4.6)*, the following hold.*

(i) If $\mu_0 > 1$, then

$$t_\Lambda^k(n) = t_\Lambda^0(n) \cdot \alpha^{-\mu_0 k} A_{-\mu_0}^k \left[\zeta_{\mathcal{R}}(\mu_0)\right]^k.$$

(ii) If $\mu_0 = 1$, then

$$t_\Lambda^k(n) = t_\Lambda^0(n) \cdot \alpha^{-k} \sum \left(\underset{t=1}{\mathrm{FP}} \{\zeta_\Lambda(t)\} - A_{-1} \ln \alpha, \zeta_\Lambda(2), 2\zeta_\Lambda(3), \ldots, (k-1)!\zeta_\Lambda(k) \right).$$

(iii) If $\mu_0 < 1$, then

$$t_\Lambda^k(n) = t_\Lambda^0(n) \cdot \alpha^{-k} \sum \left(\zeta_\Lambda(1), \zeta_\Lambda(2), 2\zeta_\Lambda(3), \ldots, (k-1)!\zeta_\Lambda(k)\right).$$

We call Theorems 4.1 and 4.2, the General Moment Theorems.

For $k = 1$, the above theorem yields the following corollary.

Corollary 4.1. *For Λ-type partitions, we have*

(i) For $\mu_0 < 1$,

$$t_\Lambda^1(n) = t_\Lambda^0(n) \cdot \frac{\zeta_\Lambda(1)}{\alpha},$$

(ii) For $\mu_0 = 1$,

$$t_\Lambda^1(n) = t_\Lambda^0(n) \cdot \frac{1}{\alpha} \left(\underset{t=1}{\mathrm{FP}} \{\zeta_\Lambda(t)\} - A_{-1} \ln \alpha \right),$$

(iii) For $\mu_0 > 1$,

$$t_\Lambda^1(n) = t_\Lambda^0(n) \cdot \frac{\zeta_{\mathcal{R}}(\mu_0) A_{-\mu_0}}{\alpha^{\mu_0}},$$

where α is the solution of (4.6) *and $t_\Lambda^0(n)$ is given in Theorem 4.1.*

4.3 Properties of Summands of Λ-Type Sequences

As in Section 3.3, in this section we give expressions for the expected number of summands as well as the variance of of the average number of summands.

Definition 4.2. The expected number of summands of a Λ-type partition of an integer n, denoted by $m_\Lambda(n)$, is

$$m_\Lambda(n) = \frac{t_\Lambda^1(n)}{t_\Lambda^0(n)}.$$

Proposition 4.2. *The expected number of summands of a* Λ*-type partition of an integer* n *is*

$$m_\Lambda(n) = S_\Lambda(e^{-\alpha})\left[1 + \mathcal{O}\left(\alpha\right)\right],$$

where α *is the solution of* (4.6).

Proof. This result is an immediate consequence of Proposition 4.1. □

A direct corollary of equations (4.19), (4.21), and (4.23), is the following result.

Theorem 4.3. *For* α *the solution of* (4.6), *the following hold.*

(i) If $\mu_0 < 1$, *then*

$$m_\Lambda(n) = \frac{\zeta_\Lambda(1)}{\alpha}\left[1 + \mathcal{O}\left(\alpha^{1-\mu_0}\right)\right]$$

(ii) If $\mu_0 = 1$, *then*

$$m_\Lambda(n) = \frac{1}{\alpha}\left(\underset{t=1}{\mathrm{FP}}\left\{\zeta_\Lambda(t)\right\} - A_{-1}\ln\alpha\right)\left[1 + \mathcal{O}\left(\alpha^{1-\mu_1}\right)\right]$$

(iii) If $\mu_0 > 1$, *then*

$$m_\Lambda(n) = \frac{\zeta_{\mathcal{R}}(\mu_0)A_{-\mu_0}}{\alpha^{\mu_0}}\left[1 + \mathcal{O}\left(\alpha^{\min(\mu_0-1,\mu_0-\mu_1)}\right)\right]$$

Definition 4.3. We define the variance, $\sigma^2_\Lambda(n)$, over the the sequence Λ as

$$\sigma^2_\Lambda(n) = \frac{t^2_\Lambda(n)}{t^0_\Lambda(n)} - \left(\frac{t^1_\Lambda(n)}{t^0_\Lambda(n)}\right)^2.$$

Let us recall Proposition 4.1, which states for all $k \in \mathbb{N}_0$ that,

$$t^k_\Lambda(n) = S^{(k)}_\Lambda(e^{-\alpha}) \cdot t^0_\Lambda(n),$$

so that Definition 4.3 gives directly

$$\sigma^2_\Lambda(n) = \left[S^{(2)}_\Lambda(e^{-\alpha}) - \left(S_\Lambda(e^{-\alpha})\right)^2\right][1 + \mathcal{O}(\alpha)].$$

We note here that (4.18) gives,

$$S^{(2)}_\Lambda(e^{-\alpha}) = \sum(S_\Lambda(e^{-\alpha}), \vartheta S_\Lambda(e^{-\alpha})),$$

where above sum is over all solutions $b_1, b_2 \in \mathbb{N}_0$ of $b_1 + 2b_2 = 2$. This yields only two solutions, $(b_1, b_2) = (2, 0), (0, 1)$, so that the above equation becomes

$$S_\Lambda^{(2)}(e^{-\alpha}) = \big(S_\Lambda(e^{-\alpha})\big)^2 + \vartheta S_\Lambda(e^{-\alpha}),$$

which gives the following lemma.

Lemma 4.2. *For a nondecreasing sequence Λ, the variance, $\sigma_\Lambda^2(n)$, is*

$$\sigma_\Lambda^2(n) = \vartheta S_\Lambda(e^{-\alpha})[1 + \mathcal{O}(\alpha)].$$

Applying (4.29), (4.28), and (4.27) to Lemma 4.2 we yield our next theorem.

Theorem 4.4. *For a nondecreasing sequence Λ, the following hold:*

(i) If $\mu_0 < 2$, then

$$\sigma_\Lambda^2(n) = \alpha^{-2}\zeta_\Lambda(2)[1 + \mathcal{O}(\alpha)].$$

(ii) If $\mu_0 = 2$, then

$$\sigma_\Lambda^2(n) = \alpha^{-2}\left[\operatorname*{FP}_{t=2}\{\zeta_\Lambda(t)\} + A_{-2}(\psi(2) - \ln\alpha) + \operatorname*{FP}_{t=2}\left\{\zeta_{\mathcal{B}}^{(2)}(t,1)\right\}\right][1 + \mathcal{O}(\alpha)].$$

(iii) If $\mu_0 > 2$, then

$$\sigma_\Lambda^2(n) = \alpha^{-2}A_{-2}\zeta_{\mathcal{B}}^{(2)}(\mu_0, 1)[1 + \mathcal{O}(\alpha)].$$

CHAPTER FIVE

Some Applications of General Moment Theorems

The thermodynamic approach to the partition problem is of considerable interest as it has led to generalizations which so far have not yielded to the methods of the analytic theory of numbers.

V.S. Nanda [20], 1954

5.1 Introduction

In this Chapter we apply the General Moment Theorems of the previous chapter to a variety of special cases. We will first examine the possibility of an abstract case of only one singularity, then proceed to give values to the singularities, which give rise to three different special types of partitions: Barnes, Epstein, and eigenvalues from PDE's. As in previous chapters, higher moments and variance are examined for each case.

5.2 Zeta-functions with One Singularity

Let us suppose that we have a nondecreasing sequence of numbers Λ, whose corresponding partition function $\Theta(t)$ admits the full asymptotic expansion,

$$\Theta(t) = \sum_{i=0}^{\infty} A_{k_i} t^{k_i},$$

in which $k_0 < 0$, and $k_i > 0$ for all $i > 0$. Define $\mu = -k_0$. Then the zeta-function associated with the sequence Λ, $\zeta_\Lambda(t)$, has only one singularity at $t = \mu$. Having met the assumptions of the General Moment Theorems, we proceed to apply them to the above sequence Λ.

Note that within the previous chapter we could not solve for n in terms of the saddle point α. Since we now have only one singularity, this is no longer a problem, and (4.6) gives

$$\alpha = \left(\frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{n}\right)^{\frac{1}{\mu+1}} \left[1 + \mathcal{O}\left(n^{\frac{-\mu}{\mu+1}}\right)\right].$$

The General Moment Theorems now yield the following corollaries.

Corollary 5.1. *For nondecreasing sequences* Λ *with associated zeta-function,* $\zeta_\Lambda(t)$, *having only one singularity at* $t = \mu$, *we have*

$$t^0_\Lambda(n) = (2\pi(\mu+1))^{-\frac{1}{2}} \left(\mu\zeta_{\mathcal{R}}(\mu+1)A_{-\mu}\right)^{\frac{1-2A_0}{2(\mu+1)}} n^{\frac{2A_0-2-\mu}{2(\mu+1)}} \times$$
$$\times \exp\left[\left(\frac{n}{\mu}\right)^{\frac{\mu}{\mu+1}} (\mu+1)\left(\zeta_{\mathcal{R}}(\mu+1)A_{-\mu}\right)^{\frac{1}{\mu+1}} + A_0 + \zeta'_\Lambda(0)\right] \times$$
$$\times \left[1 + \mathcal{O}\left(n^{\frac{-1}{\mu+1}}\right)\right].$$

Since the relationship between μ and 1 is not specified the second general moment theorem yields multiple cases.

Corollary 5.2. *For nondecreasing sequences* Λ *with associated zeta-function,* $\zeta_\Lambda(t)$, *having only one singularity at* $t = \mu$, *we have for* $k \geq 1$ *the following.*

(i) For $\mu > 1$,

$$t^k_\Lambda(n) = t^0_\Lambda(n) \cdot \left(\frac{n}{\mu\zeta_{\mathcal{R}}(\mu+1)}\right)^{\frac{\mu k}{\mu+1}} A_{-\mu}^{\frac{k}{\mu+1}} \left[\zeta_{\mathcal{R}}(\mu)\right]^k.$$

(ii) For $\mu = 1$,

$$t^k_\Lambda(n) = t^0_\Lambda(n) \cdot n^{\frac{k}{2}} \left(\frac{\pi^2 A_{-1}}{6}\right)^{\frac{-k}{2}} \times$$
$$\times \sum \left(\underset{t=1}{\mathrm{FP}}\left\{\zeta_\Lambda(t)\right\} - \frac{A_{-1}}{\mu+1}\ln\left(\frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{n}\right), \zeta_\Lambda(2), 2\zeta_\Lambda(3), \ldots, (k-1)!\zeta_\Lambda(k)\right)$$

(iii) For $\mu < 1$,

$$t^k_\Lambda(n) = t^0_\Lambda(n) \cdot n^{\frac{k}{\mu+1}} \left(\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}\right)^{\frac{-k}{\mu+1}} \sum \left(\zeta_\Lambda(1), \zeta_\Lambda(2), 2\zeta_\Lambda(3), \ldots, (k-1)!\zeta_\Lambda(k)\right).$$

Applying Theorems 4.3 and 4.3 give the following corollaries.

Corollary 5.3. *For nondecreasing sequences* Λ *with associated zeta-function,* $\zeta_\Lambda(t)$*, having only one singularity at* $t = \mu$*, the following hold.*

(i) If $\mu < 1$*, then*

$$m_\Lambda(n) = \zeta_\Lambda(1) \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}} \right)^{\frac{1}{\mu+1}} \left[1 + \mathcal{O}\left(n^{\frac{-1}{\mu+1}}\right)\right]$$

(ii) If $\mu = 1$*, then*

$$m_\Lambda(n) = \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}} \right)^{\frac{1}{\mu+1}} \times \left(\left(\underset{t=1}{\mathrm{FP}} \{\zeta_\Lambda(t)\} - \frac{A_{-1}}{\mu+1} \ln \frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{n} \right) \right) \left[1 + \mathcal{O}\left(n^{\frac{-1}{\mu+1}}\right)\right]$$

(iii) If $\mu > 1$*, then*

$$m_\Lambda(n) = \zeta_{\mathcal{R}}(\mu) A_{-\mu} \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}} \right)^{\frac{\mu}{\mu+1}} \left[1 + \mathcal{O}\left(n^{\frac{-1}{\mu+1}}\right)\right]$$

Corollary 5.4. *For nondecreasing sequences* Λ *with associated zeta-function,* $\zeta_\Lambda(t)$*, having only one singularity at* $t = \mu$*, the following hold.*

(i) If $\mu < 2$*, then*

$$\sigma^2_\Lambda(n) = n^{\frac{2}{\mu+1}} \left(\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}\right)^{\frac{-2}{\mu+1}} \zeta_\Lambda(2)[1 + \mathcal{O}\left(n^{\frac{-1}{\mu+1}}\right)].$$

(ii) If $\mu = 2$*, then*

$$\sigma^2_\Lambda(n) = n^{\frac{2}{\mu+1}} \left(\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}\right)^{\frac{-2}{\mu+1}} \times \left[\underset{t=2}{\mathrm{FP}} \{\zeta_\Lambda(t)\} + A_{-2} \left(\psi(2) - \frac{1}{\mu+1} \ln \left(\frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{n} \right) \right) + \underset{t=2}{\mathrm{FP}} \left\{ \zeta^{(2)}_{\mathcal{B}}(t,1) \right\} \right] \times [1 + \mathcal{O}\left(n^{\frac{-1}{\mu+1}}\right)].$$

(iii) If $\mu > 2$*, then*

$$\sigma^2_\Lambda(n) = n^{\frac{2}{\mu+1}} \left(\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}\right)^{\frac{-2}{\mu+1}} A_{-2} \zeta^{(2)}_{\mathcal{B}}(\mu,1)[1 + \mathcal{O}\left(n^{\frac{-1}{\mu+1}}\right)].$$

5.3 Barnes Type Moments

We will now consider two special cases of Barnes type moments, the 2-dimensional and 3-dimensional cases, with $\vec{r} = \vec{1}$ and $a = 0$. These results correspond to two and three dimensional oscillator assemblies considered by Nanda [20].

5.3.1 2-dimensional

For the 2-dimensional case, (4.6) gives

$$\alpha = \left(\frac{2\zeta_{\mathcal{R}}(3)}{n}\right)^{\frac{1}{3}} + \left(\frac{\zeta_{\mathcal{R}}(2)}{3(2\zeta_{\mathcal{R}}(3))^{\frac{1}{3}}}\right) n^{\frac{-2}{3}} - \frac{7}{36n} + \mathcal{O}\left(n^{\frac{-4}{3}}\right)$$

We can also evaluate $\zeta'_{\mathcal{B}_2}(0)$ using (2.14) to give

$$\zeta'_{\mathcal{B}_2}(0) = -\frac{1}{2}\ln 2\pi + \zeta'_{\mathcal{R}}(-1).$$

We may now apply the General Moment Theorems to yield the following corollaries.

Corollary 5.5. *For 2-dimensional Barnes Type sequence with $a = 0$ and $\vec{r} = \vec{1}$,*

$$t^0_{\mathcal{B}_2}(n) = \frac{(6\zeta_{\mathcal{R}}(2))^{\frac{-1}{2}}}{2\pi}\left(\frac{2\zeta_{\mathcal{R}}(2)}{n}\right)^{\frac{31}{36}} \times \exp\left[\frac{3(\zeta_{\mathcal{R}}(3))^{\frac{1}{3}}}{2^{\frac{2}{3}}} n^{\frac{2}{3}} + \frac{\zeta_{\mathcal{R}}(2)}{2^{\frac{1}{3}}(\zeta_{\mathcal{R}}(3))^{\frac{1}{3}}} n^{\frac{1}{3}} - \frac{(\zeta_{\mathcal{R}}(2))^2}{12\zeta_{\mathcal{R}}(3)} + \zeta'_{\mathcal{R}}(-1)\right] \times \left[1 + \mathcal{O}\left(n^{\frac{-1}{3}}\right)\right].$$

Making the substitution $n' = n/\left(2\zeta_{\mathcal{R}}(3)\right)$, for ease of comparison, the above corollary corresponds to equation (34) on p.597 of Nanda [20].

Corollary 5.6. *For 2-dimensional Barnes Type sequence with $a = 0$, $\vec{r} = \vec{1}$, and $k \geq 1$,*

$$t^k_{\mathcal{B}_2}(n) = \frac{(6\zeta_{\mathcal{R}}(2))^{\frac{-1}{2}}}{2\pi}\left(\frac{\pi^2}{6}\right)^k\left(\frac{2\zeta_{\mathcal{R}}(2)}{n}\right)^{\frac{31-24k}{36}} \times \exp\left[\frac{3(\zeta_{\mathcal{R}}(3))^{\frac{1}{3}}}{2^{\frac{2}{3}}} n^{\frac{2}{3}} + \frac{\zeta_{\mathcal{R}}(2)}{2^{\frac{1}{3}}(\zeta_{\mathcal{R}}(3))^{\frac{1}{3}}} n^{\frac{1}{3}} - \frac{(\zeta_{\mathcal{R}}(2))^2}{12\zeta_{\mathcal{R}}(3)} + \zeta'_{\mathcal{R}}(-1)\right] \times \left[1 + \mathcal{O}\left(n^{\frac{-1}{3}}\right)\right].$$

Note that the case $k = 0$ of Corollary 5.6 is precisely $t^0_{\mathcal{B}_2}(n)$ of Corollary 5.5. In general this is true when the largest singularity is greater than 1; that is, in this paper, $\mu_0 > 1$. Thus when considering the 3-dimensional Barnes case, we will give only the general result for $k \in \mathbb{N}_0$. But first we give the expected number of summands and variance as consequences of Theorems 4.3 and 4.4.

Corollary 5.7. *For 2-dimensional Barnes Type partitions with $a = 0$ and $\vec{r} = \vec{1}$,*

$$m_{\mathcal{B}_2}(n) = \zeta_{\mathcal{R}}(2)\left(\frac{n}{2\zeta_{\mathcal{R}}(3)}\right)^{\frac{2}{3}}\left[1 + \mathcal{O}\left(n^{\frac{-1}{3}}\right)\right].$$

Corollary 5.8. *For 2-dimensional Barnes Type partitions with $a = 0$ and $\vec{r} = \vec{1}$,*

$$\begin{aligned}\sigma^2_{\mathcal{B}_2}(n) = &(2\zeta_{\mathcal{R}}(3))^{\frac{-2}{3}} n^{\frac{2}{3}} \\ &\times\left[\operatorname*{FP}_{t=2}\left\{\zeta^{(2)}_{\mathcal{B}}(t)\right\} + \psi(2) - \frac{1}{3}\ln\left(\frac{2\zeta_{\mathcal{R}}(3)}{n}\right) + \operatorname*{FP}_{t=2}\left\{\zeta^{(2)}_{\mathcal{B}}(t,1)\right\}\right] \\ &\times\left[1 + \mathcal{O}\left(n^{\frac{-1}{3}}\right)\right].\end{aligned}$$

5.3.2 3-dimensional

For the 3-dimensional case, we evaluate (4.6) to give

$$\begin{aligned}\alpha = &(3\zeta_{\mathcal{R}}(4))^{\frac{1}{4}} n^{\frac{-1}{4}} + \left(\frac{3^{\frac{1}{2}}\zeta_{\mathcal{R}}(3)}{4(\zeta_{\mathcal{R}}(4))^{\frac{1}{2}}}\right) n^{\frac{-1}{2}} \\ &+ \left(\frac{8\zeta_{\mathcal{R}}(2)\zeta_{\mathcal{R}}(4) - 3(\zeta_{\mathcal{R}}(3))^2}{3^{\frac{1}{4}}32(\zeta_{\mathcal{R}}(4))^{\frac{5}{4}}}\right) n^{\frac{-3}{4}} - \frac{5}{32n} + \mathcal{O}\left(n^{\frac{-5}{4}}\right).\end{aligned}$$

As before we evaluate $\zeta'_{\mathcal{B}_3}(0)$. Using (2.14)

$$\zeta'_{\mathcal{B}_3}(0) = -\frac{1}{2}\ln 2\pi + \frac{3}{2}\zeta'_{\mathcal{R}}(-1) + \frac{1}{2}\zeta'_{\mathcal{R}}(-2).$$

We apply the General Moment Theorems to yield the following corollaries.

Corollary 5.9. *For 3-dimensional Barnes Type sequence with $a = 0$, $\vec{r} = \vec{1}$, and $k \in \mathbb{N}_0$,*

$$t^k_{\mathcal{B}_3}(n) = \frac{(3\zeta_{\mathcal{R}}(4))^{\frac{-1}{2}}}{4\pi} (\zeta_{\mathcal{R}}(3))^k \left(\frac{3\zeta_{\mathcal{R}}(4)}{n}\right)^{\frac{25-24k}{32}} \times \exp\left[4\zeta_{\mathcal{R}}(4)\left(\frac{n}{3\zeta_{\mathcal{R}}(4)}\right)^{\frac{3}{4}} + \frac{3\zeta_{\mathcal{R}}(3)}{2}\left(\frac{n}{3\zeta_{\mathcal{R}}(4)}\right)^{\frac{1}{2}} + \right. \left. + \left(\zeta_{\mathcal{R}}(2) - \frac{3(\zeta_{\mathcal{R}}(3))^2}{8\zeta_{\mathcal{R}}(4)}\right)\left(\frac{n}{3\zeta_{\mathcal{R}}(4)}\right)^{\frac{1}{4}} + C\right]\left[1 + \mathcal{O}\left(n^{\frac{-1}{4}}\right)\right],$$

where $C = \frac{(\zeta_{\mathcal{R}}(3))^3}{8(\zeta_{\mathcal{R}}(4))2} - \frac{\zeta_{\mathcal{R}}(2)\zeta_{\mathcal{R}}(3)}{4\zeta_{\mathcal{R}}(4)} + \frac{3}{2}\zeta'_{\mathcal{R}}(-1) + \frac{1}{2}\zeta'_{\mathcal{R}}(-2)$.

Setting $k = 0$, and making the substitution $n" = n/(3\zeta_{\mathcal{R}}(4))$, the above corollary corresponds to equation (51) on p.599 of Nanda [20].

Now, Theorems 4.3 and 4.4 yield the following corollaries.

Corollary 5.10. *For 3-dimensional Barnes Type partitions with $a = 0$ and $\vec{r} = \vec{1}$,*

$$m_{\mathcal{B}_3}(n) = \zeta_{\mathcal{R}}(3)\left(\frac{n}{3\zeta_{\mathcal{R}}(4)}\right)^{\frac{3}{4}}\left[1 + \mathcal{O}\left(n^{\frac{-1}{4}}\right)\right].$$

Corollary 5.11. *For 3-dimensional Barnes Type sequence with $a = 0$, $\vec{r} = \vec{1}$,*

$$\sigma^2_{\mathcal{B}_3}(n) = (3\zeta_{\mathcal{R}}(4))^{\frac{-1}{2}} n^{\frac{1}{2}} \zeta^{(2)}_{\mathcal{B}}(3,1)\left[1 + \mathcal{O}\left(n^{\frac{-1}{4}}\right)\right].$$

5.4 Epstein Type Moments

Let us recall, that for a Dirichlet boundary condition, as $t \to 0$ the partition function of the Epstein zeta-function, $\Theta_{\mathcal{E}}(t)$, admits to the asymptotic expansion (see (2.22))

$$\Theta_{\mathcal{E}}(t) \sim \sum_{n=0}^{d} \frac{c_n}{2^d} t^{\frac{-n}{2}} \tag{5.1}$$

where the c_i's are as in (2.23).

We now approach 2 and 3-dimensional Epstein Type moments in the same way that we considered the Barnes type.

5.4.1 2-dimensional

Note that for the 2-dimensional case, (4.6) gives

$$\alpha = \left(\frac{\pi\zeta_{\mathcal{R}}(2)}{4}\right)^{\frac{1}{2}} n^{\frac{-1}{2}} - \left(\frac{\pi^{\frac{1}{4}}\zeta_{\mathcal{R}}\left(\frac{3}{2}\right)}{4\sqrt{2}(\zeta_{\mathcal{R}}(2))^{\frac{1}{4}}}\right) n^{\frac{-3}{4}} + \frac{1}{8n} + \mathcal{O}\left(n^{\frac{-5}{4}}\right).$$

Using this information we gain the following corollaries to the General Moment Theorems.

Corollary 5.12. *For 2-dimensional Epstein Type sequence with $a = 0$ and $\vec{r} = \vec{1}$,*

$$t^0_{\mathcal{E}_2}(n) = \left(\pi^2\zeta_{\mathcal{R}}(2)\right)^{\frac{-1}{2}} \left(\frac{\pi\zeta_{\mathcal{R}}(2)}{4n}\right)^{\frac{5}{8}} \times \exp\left[\sqrt{\pi\zeta_{\mathcal{R}}(2)n} - \left(\frac{\pi^{\frac{1}{4}}\zeta_{\mathcal{R}}\left(\frac{3}{2}\right)}{\sqrt{2}(\zeta_{\mathcal{R}}(2))^{\frac{1}{4}}}\right) n^{\frac{1}{4}} - \frac{\left(\zeta_{\mathcal{R}}\left(\frac{3}{2}\right)\right)^2}{16\zeta_{\mathcal{R}}(2)} + \zeta'_{\mathcal{E}_2}(0)\right] \times \left[1 + \mathcal{O}\left(n^{\frac{-1}{4}}\right)\right].$$

Corollary 5.13. *For 2-dimensional Epstein Type sequence with $a = 0$, $\vec{r} = \vec{1}$, and $k \geq 1$,*

$$t^k_{\mathcal{E}_2}(n) = \left(\pi^2\zeta_{\mathcal{R}}(2)\right)^{\frac{-1}{2}} \left(\frac{\pi\zeta_{\mathcal{R}}(2)}{4n}\right)^{\frac{5-4k}{8}} \times \exp\left[\sqrt{\pi\zeta_{\mathcal{R}}(2)n} - \left(\frac{\pi^{\frac{1}{4}}\zeta_{\mathcal{R}}\left(\frac{3}{2}\right)}{\sqrt{2}(\zeta_{\mathcal{R}}(2))^{\frac{1}{4}}}\right) n^{\frac{1}{4}} - \frac{\left(\zeta_{\mathcal{R}}\left(\frac{3}{2}\right)\right)^2}{16\zeta_{\mathcal{R}}(2)} + \zeta'_{\mathcal{E}_2}(0)\right]$$
$$\times \sum\left(\underset{t=1}{\mathrm{FP}}\{\zeta_{\mathcal{E}_2}(t)\} - \frac{\pi}{8}\ln\left(\frac{\pi\zeta_{\mathcal{R}}(2)}{4n}\right), \zeta_{\mathcal{E}_2}(2), 2\zeta_{\mathcal{E}_2}(3), \ldots, (k-1)!\zeta_{\mathcal{E}_2}(k)\right) \times \left[1 + \mathcal{O}\left(n^{\frac{-1}{4}}\right)\right].$$

Theorems 4.3 and 4.4 give the expected number of summands and the variance of summands.

Corollary 5.14. *For 2-dimensional Epstein Type sequence with $a = 0$ and $\vec{r} = \vec{1}$,*

$$m_{\mathcal{E}_2}(n) = \left(\frac{4n}{\pi\zeta_{\mathcal{R}}(2)}\right)^{\frac{1}{2}} \left(\underset{t=1}{\mathrm{FP}}\left\{\zeta^{(2)}_{\mathcal{E}}(t)\right\} - \frac{\pi}{8}\ln\left(\frac{\pi\zeta_{\mathcal{R}}(2)}{4n}\right)\right) \left[1 + \mathcal{O}\left(n^{\frac{-1}{4}}\right)\right].$$

Corollary 5.15. *For 2-dimensional Epstein Type sequence with $a = 0$ and $\vec{r} = \vec{1}$,*

$$\sigma^2_{\mathcal{E}_2}(n) = \frac{4n}{\pi\zeta_{\mathcal{R}}(2)}\zeta^{(2)}_{\mathcal{E}}(2)\left[1 + \mathcal{O}\left(n^{\frac{-1}{2}}\right)\right].$$

5.4.2 3-dimensional

For the 3-dimensional Epstein case, (4.6) gives

$$\alpha = \left(\frac{3\pi^{\frac{3}{2}}\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)}{16}\right)^{\frac{2}{5}} n^{\frac{-2}{5}} - \left(\frac{3^{\frac{3}{5}}\zeta_{\mathcal{R}}(2)}{5}\right)\left(\frac{\pi}{2\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)}\right)^{\frac{2}{5}} n^{\frac{-3}{5}}$$
$$+ \frac{3^{\frac{4}{5}}\pi^{\frac{1}{5}}}{2^{\frac{1}{5}}100\left(\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)\right)^{\frac{6}{5}}}\left(5\zeta_{\mathcal{R}}\left(\frac{3}{2}\right)\zeta_{\mathcal{R}}\left(\frac{5}{2}\right) - 2(\zeta_{\mathcal{R}}(2))^2\right) n^{\frac{-4}{5}} + \mathcal{O}\left(n^{-1}\right).$$

Here we may note that $\mu_0 = \frac{3}{2} > 1$, so that we have the following corollary to the General Moment Theorems.

Corollary 5.16. *For 3-dimensional Epstein Type sequence with $a = 0$, $\vec{r} = \vec{1}$, and $k \in \mathbb{N}_0$,*

$$t^k_{\mathcal{E}_3}(n) = \left(\frac{15}{16}\pi^{\frac{5}{2}}\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)\right)^{\frac{-1}{2}}\left(\frac{3\pi^{\frac{3}{2}}\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)}{16n}\right)^{\frac{15-4k}{20}}\left(\frac{\pi^{\frac{3}{2}}}{8}\zeta_{\mathcal{R}}\left(\frac{3}{2}\right)\right)^{k}$$
$$\times \exp\left[\left(\frac{5\pi^{\frac{3}{5}}\left(\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)\right)^{\frac{2}{5}}}{6^{\frac{3}{5}}2}\right) n^{\frac{3}{5}} - \frac{3^{\frac{3}{5}}\zeta_{\mathcal{R}}(2)}{2}\left(\frac{\pi}{2\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)}\right)^{\frac{2}{5}} n^{\frac{2}{5}}\right.$$
$$-\frac{3^{\frac{4}{5}}\pi^{\frac{1}{5}}}{2^{\frac{1}{5}}20\left(\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)\right)^{\frac{6}{5}}}\left(5\zeta_{\mathcal{R}}\left(\frac{3}{2}\right)\zeta_{\mathcal{R}}\left(\frac{5}{2}\right) - 2(\zeta_{\mathcal{R}}(2))^2\right) n^{\frac{1}{5}}$$
$$\left. -\frac{(\zeta_{\mathcal{R}}(2))^3}{10\left(\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)\right)^2} + \frac{3\zeta_{\mathcal{R}}\left(\frac{3}{2}\right)\zeta_{\mathcal{R}}(2)}{20\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)} + \zeta'_{\mathcal{E}_3}(0)\right]\left[1 + \mathcal{O}\left(n^{\frac{-1}{5}}\right)\right].$$

Theorems 4.3 and 4.4 yield the following corollaries.

Corollary 5.17. *For 3-dimensional Epstein Type sequence with $a = 0$ and $\vec{r} = \vec{1}$,*

$$m_{\mathcal{E}_3}(n) = \frac{\pi^{\frac{3}{2}}\zeta_{\mathcal{R}}\left(\frac{3}{2}\right)}{8}\left(\frac{16n}{3\pi^{\frac{3}{2}}\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)}\right)^{\frac{3}{5}}\left[1 + \mathcal{O}\left(n^{\frac{-1}{5}}\right)\right].$$

Corollary 5.18. *For 3-dimensional Epstein Type sequence with $a = 0$, $\vec{r} = \vec{1}$,*

$$\sigma^2_{\mathcal{E}_3}(n) = \left(\frac{16n}{3\pi^{\frac{3}{5}}\zeta_{\mathcal{R}}\left(\frac{5}{2}\right)}\right)^{\frac{4}{5}}\zeta^{(3)}_{\mathcal{E}}(2)\left[1 + \mathcal{O}\left(n^{\frac{-1}{5}}\right)\right].$$

5.5 Moments over Spectra from PDE's

As a preliminary to considering a general partial differential equation, we first take for an example the 2-dimensional Epstein zeta-function with Neumann boundary conditions, $a = 0$, and $\vec{r} = \vec{1}$. Note that from an earlier section we have

$$\sum_{n_1,n_2\in\mathbb{N}_0} e^{-(n_1^2+n_2^2)t} \sim \frac{\pi}{4}t^{-1} + \frac{\sqrt{\pi}}{2}t^{\frac{-1}{2}} + \frac{1}{4}. \tag{5.2}$$

The above partition function arises naturally when one considers the manifold $\mathcal{M}$ in Figure 5.1 below.

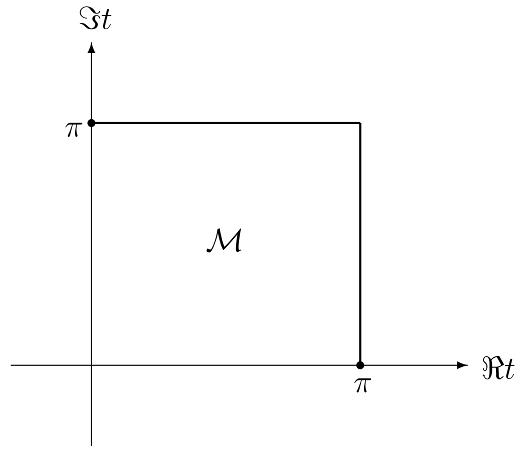

Figure 5.1. The Manifold $\mathcal{M}$ in the Complex Plane.

Note that $\mathrm{vol}(\mathcal{M}) = \pi^2$ and $\mathrm{vol}(\partial\mathcal{M}) = 4\pi$. Also note (5.2) gives

$$A_{-1} = \frac{\pi}{4} = (4\pi)^{-1}\mathrm{vol}(\mathcal{M}) \qquad \text{and} \qquad A_{-\frac{1}{2}} = \frac{\sqrt{\pi}}{2} = \frac{1}{4}(4\pi)^{-\frac{1}{2}}\mathrm{vol}(\partial\mathcal{M}). \tag{5.3}$$

Though this seems somewhat coincidental, the following theorem realizes this general phenomenon, see Gilkey [9], Greiner [11, 12], and Seeley [27, 28, 29].

Theorem 5.1. *Let Ω be a smooth compact subset of $\mathbb{R}^d$. Let $\mathcal{P} = -\Delta + E$ where Δ is a Laplacian and $E \in \mathcal{C}^\infty(\Omega)$. Denote λ_n as the eigenvalues such that $\mathcal{P}\varphi_n = \lambda_n\varphi_n$ where φ_n has Neumann boundary conditions. Then*

$$\Theta(t) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \sim \sum_{l=0}^{\infty} A_{\frac{l-d}{2}} t^{\frac{l-d}{2}}. \tag{5.4}$$

The first five coefficients, the A_i's, of the above sequence are known, see [18] pp.166-178 for details. The first two coefficients are easy to work with. They are

$$A_{\frac{-d}{2}} = (4\pi)^{\frac{-d}{2}}\mathrm{vol}(\Omega)$$
$$A_{\frac{-(d-1)}{2}} = (4\pi)^{\frac{-(d-1)}{2}}\frac{1}{4}\mathrm{vol}(\partial\Omega).$$

Though the next three coefficients are known, they involve much more complication in calculation, and are not paramount to our discussion.

The above theorem gives a full asymptotic expansion of the partition function so that we may apply the General Moment Theorems to the sequence of eigenvalues. Note that (4.6) gives

$$n = \frac{d\zeta_{\mathcal{R}}(\frac{d}{2}+1)}{2\alpha^{\frac{d}{2}+1}}A_{-\frac{d}{2}} + \frac{(d-1)\zeta_{\mathcal{R}}(\frac{d-1}{2}+1)}{2\alpha^{\frac{d-1}{2}+1}}A_{-\frac{(d-1)}{2}} + \sum_{i=2}^{d}\frac{(d-i)\zeta_{\mathcal{R}}(\frac{l-d}{2}+1)}{2\alpha^{\frac{d-i}{2}+1}}A_{\frac{i-d}{2}} + \frac{A_0}{\alpha} + \frac{A_{1/2}}{2} + \frac{A_1}{6}\alpha + \mathcal{O}(\alpha^3). \quad (5.5)$$

In general an equation relating α in terms of n can be found if desired, we leave this to the interested reader. Though our result will not be as accurate as previous results, we will assume

$$\alpha \approx \left(\frac{2^{d+1}\pi^{\frac{d}{2}}n}{d\zeta_{\mathcal{R}}\left(\frac{d}{2}+1\right)\mathrm{vol}(\Omega)}\right)^{\frac{2}{d+2}}, \quad (5.6)$$

using only the leading order of the series.

The first general moment theorem gives the following corollary.

Corollary 5.19. *For the sequence of eigenvalues of a partial differential equation over Ω a smooth compact subset of $\mathbb{R}^d$,*

$$t^0_{PDE}(n) = \left(\frac{d(d+2)}{2^{d+2}\pi^{\frac{d-2}{2}}}\zeta_{\mathcal{R}}\left(\frac{d}{2}+1\right)\mathrm{vol}(\Omega)\right)^{-\frac{1}{2}}\left(\frac{2^{d+1}\pi^{\frac{d}{2}}n}{d\zeta_{\mathcal{R}}\left(\frac{d}{2}+1\right)\mathrm{vol}(\Omega)}\right)^{\frac{d-4+4A_0}{2(d+2)}}$$
$$\times\exp\left[\sum_{i=0}^{d}A_{-\frac{d-i}{2}}\left(\frac{d-i+2}{2}\right)\zeta_{\mathcal{R}}\left(\frac{d-i+2}{2}\right)\left(\frac{2^{d+1}\pi^{\frac{d}{2}}n}{d\zeta_{\mathcal{R}}\left(\frac{d}{2}+1\right)\mathrm{vol}(\Omega)}\right)^{\frac{i-d}{d+2}} + A_0 + \zeta'_{PDE}(0)\right]\left[1+\mathcal{O}\left(n^{\frac{-1}{d+2}}\right)\right]. \quad (5.7)$$

If the dimension d is given, it is quite easy to apply the second general moment theorem, Theorem 4.2. Since our consideration does not specify a certain dimension d, Theorem 4.2 does not contribute anymore information than as previously stated, thus we do not consider the case of higher moments at this time.

A much more pleasant calculation is that of the expected number of summands and the variance. Theorem 4.3 and 4.4 give the following corollary for the partition over eigenvalues.

Corollary 5.20. *For the partition over eigenvalues of a partial differential equation over Ω a smooth compact subset of $\mathbb{R}^d$, the following hold.*

(i) If $d = 1$, then

$$m_{PDE}(n) = \zeta_{PDE}(1) \left(\frac{4n\sqrt{\pi}}{\zeta_{\mathcal{R}}\left(\frac{3}{2}\right) \operatorname{vol}(\Omega)} \right)^{\frac{2}{3}} \left[1 + \mathcal{O}\left(n^{\frac{-1}{3}} \right) \right].$$

(ii) If $d = 2$, then

$$m_{PDE}(n) = \left(\frac{4n\sqrt{\pi}}{2\zeta_{\mathcal{R}}(2)\operatorname{vol}(\Omega)} \right)^{\frac{1}{2}} \times \left(\underset{t=1}{\operatorname{FP}} \left\{ \zeta_{PDE}(t) \right\} - \frac{\operatorname{vol}(\Omega)}{8\pi} \ln\left(\frac{2\zeta_{\mathcal{R}}(2)\operatorname{vol}(\Omega)}{4n\sqrt{\pi}} \right) \right) \left[1 + \mathcal{O}\left(n^{\frac{-1}{4}} \right) \right].$$

(iii) If $d \geq 3$, then

$$m_{PDE}(n) = \frac{\zeta_{\mathcal{R}}\left(\frac{d}{2}\right) \operatorname{vol}(\Omega)}{(4\pi)^{\frac{d}{2}}} \left(\frac{2n(4\pi)^{\frac{d}{2}}}{d\zeta_{\mathcal{R}}\left(\frac{d}{2}+1\right) \operatorname{vol}(\Omega)} \right)^{\frac{d}{d+2}} \left[1 + \mathcal{O}\left(n^{\frac{-d}{2(d+2)}} \right) \right].$$

Corollary 5.21. *For the partition over eigenvalues of a partial differential equation over Ω a smooth compact subset of $\mathbb{R}^d$, the following hold.*

(i) For $d < 4$,

$$\sigma^2_{PDE}(n) = n^{\frac{4}{d+2}} \left(\frac{2^{d+1}\pi^{\frac{d}{2}} n}{d\zeta_{\mathcal{R}}\left(\frac{d}{2}+1\right) \operatorname{vol}(\Omega)} \right)^{\frac{4}{d+2}} \zeta_{PDE}(2) \left[1 + \mathcal{O}\left(n^{\frac{-1}{d+2}} \right) \right].$$

(ii) For $d = 4$,

$$\sigma^2_{PDE}(n) = n^{\frac{4}{d+2}} \left(\frac{2^{d+1}\pi^{\frac{d}{2}} n}{d\zeta_{\mathcal{R}}\left(\frac{d}{2}+1\right)\mathrm{vol}(\Omega)} \right)^{\frac{4}{d+2}} \times \left[\underset{t=2}{\mathrm{FP}} \left\{ \zeta_{PDE}(t) \right\} + \frac{\mathrm{vol}(\Omega)}{(4\pi)^{\frac{d}{2}}} \left(\psi(2) + \frac{2}{d+2} \ln \left(\frac{2^{d+1}\pi^{\frac{d}{2}} n}{d\zeta_{\mathcal{R}}\left(\frac{d}{2}+1\right)\mathrm{vol}(\Omega)} \right) \right) + \underset{t=2}{\mathrm{FP}} \left\{ \zeta^{(2)}_{\mathcal{B}}(t,2) \right\} \right] \left[1 + \mathcal{O}\left(n^{\frac{-1}{d+2}} \right) \right].$$

(iii) For $d > 4$,

$$\sigma^2_{PDE}(n) = n^{\frac{4}{d+2}} \left(\frac{2^{d+1}\pi^{\frac{d}{2}} n}{d\zeta_{\mathcal{R}}\left(\frac{d}{2}+1\right)\mathrm{vol}(\Omega)} \right)^{\frac{4}{d+2}} A_{-2}\zeta^{(2)}_{\mathcal{B}}\left(\frac{d}{2}, 2 \right) \left[1 + \mathcal{O}\left(n^{\frac{-1}{d+2}} \right) \right],$$

where A_{-2} is, in general, not known.

CHAPTER SIX

Conclusion

This paper has presented the General Moment Theorems, general theorems for the moments of partitions over sequences whose associated zeta-functions are of arbitrary dimension, as well as a general theorem for the variance of summands. In particular, the first general moment theorem gives the number of ways to write an integer n as the sum of summands from a given sequence Λ of nondecreasing natural numbers.

After the consideration of a general one-dimensional case, applications to Barnes and Epstein type sequences were given, which led nicely into an application to sequences of eigenvalues of a partial differential equation.

Though it was not emphasized, all of the partitions discussed were unrestricted and non-distinct; that is, there was neither a restriction on the value of an element nor the number of times an element of the general sequence could be used in the partitioning. The underlying reason for the examination of unrestricted non-distinct partitions is their physical application in statistical mechanics, particularly to Bose-Einstein gases.

A natural extension of the General Moment Theorems would be to consider distinct partitions. Distinct partitions also have physical application, for example, within the consideration of Fermi-Dirac gases. Because of the similarity of the generating function used for distinct partitions, an approach similar to the one used in this paper is a likely starting point.

Restricted non-distinct partitions would be another natural extension of the General Moment Theorems.

Though a physical application is not apparent, another interesting direction to pursue, would be to examine a partitioning over a recursively defined sequence, such as the Fibonacci sequence.

As is evident from the above discussion, the ideas presented in this paper lead to many questions, which we hope to consider in the future.

APPENDICES

APPENDIX A

k-independence of Saddle-Point

A.1 k-independence of Saddle-Point Leading Order

The saddle point α is defined as the solution of

$$n = \sum_{\lambda \in \Lambda} \frac{\lambda}{e^{\alpha\lambda} - 1} - \frac{\frac{d}{da} S_\Lambda^{(k)}(e^{-\alpha})}{S_\Lambda^{(k)}(e^{-\alpha})}, \tag{A.1}$$

where $S_\Lambda^{(k)}(e^{-\alpha}) = \sum(S_\Lambda, \vartheta S_\Lambda, \ldots, \vartheta^{k-1} S_\Lambda)$.

The case $k = 0$ is considered in the text, with the note that, for leading order, this is the only case that need be considered. We set out now to prove this.

Consider now the case $k = 1$. For $\sigma > s, \mu$ and $|\mathrm{ph}\ \alpha| \leq \frac{\pi}{2} - \delta$, we have

$$S_\Lambda(e^{-\alpha}) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \alpha^{-t} \zeta_\Lambda(t) \zeta_{\mathcal{R}}(t) \Gamma(t) dt, \tag{A.2}$$

so that

$$\frac{d}{d\alpha} S_\Lambda(e^{-\alpha}) = \frac{-1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} t \alpha^{-t-1} \zeta_\Lambda(t) \zeta_{\mathcal{R}}(t) \Gamma(t) dt. \tag{A.3}$$

As we did in calculating the moments at $k = 1$, we must again consider the three cases: $\mu < 1$, $\mu = 1$, and $\mu > 1$. Direct calculation gives the following lemma.

Lemma A.1. *For $\mu > 0$,*

$$-\frac{\frac{d}{da} S_\Lambda(e^{-\alpha})}{S_\Lambda(e^{-\alpha})} = \mathcal{O}\left(\alpha^{-1}\right). \tag{A.4}$$

Proof. Let $\mu = 1$, then (4.21) gives

$$S_\Lambda(e^{-\alpha}) = \frac{1}{\alpha} \left(\underset{t=1}{\mathrm{FP}} \{\zeta_\Lambda(t)\} - A_{-1} \ln \alpha \right) [1 + \mathcal{O}(\alpha^{1-\mu_1})],$$

and similar calculation gives

$$\frac{d}{d\alpha} S_\Lambda(e^{-\alpha}) = \frac{-1}{\alpha^2} \left(\underset{t=1}{\mathrm{FP}} \{\zeta_\Lambda(t)\} - A_{-1} \ln \alpha \right) [1 + \mathcal{O}(\alpha^{1-\mu_1})].$$

Thus

$$-\frac{\frac{d}{d\alpha}S_\Lambda(e^{-\alpha})}{S_\Lambda(e^{-\alpha})} = \frac{1}{\alpha} + \mathcal{O}\left(\frac{1}{\alpha}\ln\frac{1}{\alpha}\right) = \mathcal{O}\left(\alpha^{-1}\right),$$

which is the desired result.

Since the calculations are analogous for the three cases listed above, we leave the other cases to the reader. □

Having considered this case, we now let $k \geq 2$. For this consideration we will need to take the derivative with respect to α of the sum $S_\Lambda^{(k)}(e^{-\alpha}) = \sum(S_\Lambda, \vartheta S_\Lambda, \ldots, \vartheta^{k-1}S_\Lambda)$. Direct calculation gives

$$\frac{d}{d\alpha}S_\Lambda^{(k)}(e^{-\alpha}) = \sum \frac{k!}{b_1!b_2!\cdots b_k!}[b_1\left(\frac{S_\Lambda(e^{-\alpha})}{1!}\right)^{b_1-1}\frac{d}{d\alpha}S_\Lambda(e^{-\alpha})\left(\frac{\vartheta^1 S_\Lambda(e^{-\alpha})}{2!}\right)^{b_2}\cdots +$$
$$\cdots + \frac{b_k}{k!}\left(\frac{S_\Lambda(e^{-\alpha})}{1!}\right)^{b_1}\left(\frac{\vartheta^{k-1}S_\Lambda(e^{-\alpha})}{k!}\right)^{b_k-1}\frac{d}{d\alpha}\vartheta^{k-1}S_\Lambda(e^{-\alpha})],$$

where the summation is over all solutions in $b_1, b_2, \ldots, b_m \in \mathbb{N}_0$ of $b_1+2b_2+\cdots+kb_k = k$. Note that $b_1 = k$ gives the the leading term of the above equation as well as that of $S_\Lambda^{(k)}(e^{-\alpha})$. With this in mind we proceed with the following lemma.

Lemma A.2. *For $\mu > 0$,*

$$-\frac{\frac{d}{d\alpha}S_\Lambda^{(k)}(e^{-\alpha})}{S_\Lambda^{(k)}(e^{-\alpha})} = \mathcal{O}\left(\alpha^{-1}\right). \tag{A.5}$$

Proof. From the above discussion we see that $b_1 = k$ gives the leading terms of both $\frac{d}{d\alpha}S_\Lambda^{(k)}(e^{-\alpha})$ and $S_\Lambda^{(k)}(e^{-\alpha})$. Calculating with only leading term in mind we have

$$\begin{aligned}
-\frac{\frac{d}{d\alpha}S_\Lambda^{(k)}(e^{-\alpha})}{S_\Lambda^{(k)}(e^{-\alpha})} &= -\frac{k(S_\Lambda(e^{-\alpha}))^{k-1}\frac{d}{d\alpha}S_\Lambda(e^{-\alpha}) + \cdots}{(S_\Lambda(e^{-\alpha}))^k + \cdots} \\
&= -\frac{k\frac{d}{d\alpha}S_\Lambda(e^{-\alpha})}{S_\Lambda(e^{-\alpha})} + \cdots \\
&= \mathcal{O}\left(\alpha^{-1}\right),
\end{aligned}$$

where the last equality follows from the proceeding lemma. □

These two lemmas now give us what we need to show the following proposition.

Proposition A.1. *The leading order of the saddle-point α is independent of k. Moreover, for leading order of α one need only consider the case $k = 0$.*

Proof. Let $k \geq 0$ and α be the solution of (A.1), then Lemma 6.1 and Lemma 6.2 give

$$n = \sum_{\lambda \in \Lambda} \frac{\lambda}{e^{\alpha\lambda} - 1} + \mathcal{O}\left(\alpha^{-1}\right).$$

Eq. (4.6) gives

$$n = \frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{\alpha^{\mu+1}} + \mathcal{O}\left(\alpha^{-\mu_1-1}\right) + \mathcal{O}\left(\alpha^{-1}\right),$$

but $\mathcal{O}\left(\alpha^{-\mu_1-1}\right) + \mathcal{O}\left(\alpha^{-1}\right) = \mathcal{O}\left(\alpha^{-\mu_1-1}\right)$, so that for all k

$$n = \frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{\alpha^{\mu+1}} \left[1 + \mathcal{O}\left(\alpha^{\mu-\mu_1}\right)\right],$$

which is the assertion. Hence for leading order of α one need only consider the case $k = 0$. □

A.2 k-independence of exponential contribution

For the purposes of this calculation we consider only leading orders and those terms which are dependent on k. Also we denote $\mu_0 \equiv \mu$ for convenience. Under these special terms, for $c(k)$ some k-dependent coefficient, (4.6) gives

$$\begin{aligned} n &= \frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{\alpha^{\mu+1}} + \cdots + \frac{c(k)}{\alpha} + \cdots \\ &= \frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{\alpha^{\mu+1}} \left(1 + \cdots + \frac{c(k)}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\alpha^{\mu}\right). \end{aligned}$$

Solving for $\frac{1}{\alpha}$ gives

$$\begin{aligned} \frac{1}{\alpha} &= \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\right)^{\frac{1}{\mu+1}} \frac{1}{\left(1 + \cdots + \frac{c(k)}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\alpha^{\mu}\right)^{\frac{1}{\mu+1}}} \\ &= \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\right)^{\frac{1}{\mu+1}} \left(1 + \cdots - \frac{c(k)}{(\mu+1)\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\alpha^{\mu} + \cdots\right). \end{aligned}$$

This equation yields $\alpha \cong \left(\frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{n}\right)^{\frac{1}{\mu+1}}$, so that

$$\frac{1}{\alpha} \cong \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\right)^{\frac{1}{\mu+1}} \times \tag{A.6}$$

$$\times \left(1 + \cdots - \frac{c(k)}{(\mu+1)\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}} \left(\frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{n}\right)^{\frac{\mu}{\mu+1}} + \cdots\right) \tag{A.7}$$

$$= \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\right)^{\frac{1}{\mu+1}} + \cdots \tag{A.8}$$

$$- \frac{c(k)}{(\mu+1)\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}} \left(\frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{n}\right)^{\frac{\mu-1}{\mu+1}} + \cdots . \tag{A.9}$$

To see if $c(k)$ will contribute to the moments we inspect $n\alpha_k(n)+\ln G_\Lambda(e^{-\alpha_k(n)})$, denoted by E_{α_k} for brevity, where $a_k(n)$ encompasses the above k-dependence of n. Upon inspection we have the following set of equalities.

$$n\alpha_k(n) + \ln G_\Lambda(e^{-\alpha_k(n)}) = E_{\alpha_k} = c(k) + \frac{(\mu+1)\zeta_{\mathcal{R}}(\mu+1)A_{-\mu}}{\alpha_k^{\mu}(n)} + \cdots .$$

Applying (A.8)-(A.9) to this equation yields

$$\begin{aligned}
E_{\alpha_k} &= c(k) + (\mu+1)\zeta_{\mathcal{R}}(\mu+1)A_{-\mu}[\left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\right)^{\frac{1}{\mu+1}} + \cdots \\
&\quad - \frac{c(k)}{(\mu+1)\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}} \left(\frac{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}{n}\right)^{\frac{\mu-1}{\mu+1}} + \cdots]^{\mu} \\
&= c(k) + (\mu+1)\zeta_{\mathcal{R}}(\mu+1)A_{-\mu} \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\right)^{\frac{\mu}{\mu+1}} \times \\
&\quad \times \left[1 + \cdots - \frac{c(k)}{(\mu+1)\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}} \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\right)^{\frac{-\mu}{\mu+1}} + \cdots\right]^{\mu} \\
&= c(k) + (\mu+1)\zeta_{\mathcal{R}}(\mu+1)A_{-\mu} \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\right)^{\frac{\mu}{\mu+1}} \times \\
&\quad \times \left[-\frac{c(k)\cdot\mu}{(\mu+1)\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}} \left(\frac{n}{\zeta_{\mathcal{R}}(\mu+1)\mu A_{-\mu}}\right)^{\frac{-\mu}{\mu+1}} + \cdots\right] \\
&= c(k) - c(k) + \cdots .
\end{aligned}$$

Thus the exponential contribution of the saddle-point is k-independent up to $\mathcal{O}(\alpha)$.

APPENDIX B

Fourier Analysis

This appendix contains Propositions and Examples (calculations) regarding Fourier Analysis (Transforms). It is provided for those students whom may not have seen these results before.

Example B.1. Let $F(x) = e^{-\pi x^2}$. Then $\hat{F}(x) = e^{-\pi u^2}$.

Proof. Let $F(x) = e^{-\pi x^2}$. Since $\int_{-\infty}^{\infty} |e^{-\pi x^2}| dx < \infty$, we have

$$\begin{aligned}
\hat{F}(u) &= \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x u} dx \\
&= \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x u} dx \\
&= \int_{-\infty}^{\infty} e^{-\pi x^2 - 2\pi i x u + \pi u^2 - \pi u^2} dx \\
&= \int_{-\infty}^{\infty} e^{-\pi (x+iu)^2 - \pi u^2} dx \\
&= e^{-\pi u^2} \int_{-\infty}^{\infty} e^{-\pi (x+iu)^2} dx \\
&= e^{-\pi u^2} \int_{-\infty}^{\infty} e^{-[\sqrt{\pi}(x+iu)]^2} dx.
\end{aligned}$$

Now let $v = \sqrt{\pi}(x + iu)$, so that $dv = \sqrt{\pi} dx$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-[\sqrt{\pi}(x+iu)]^2} dx &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-v^2} dv \\
&= \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1,
\end{aligned}$$

so that $\hat{F}(u) = e^{-\pi u^2}$. □

Example B.2. Let $F(x) = e^{-\pi(x+a)^2}$. Then $\hat{F}(x) = e^{2\pi iau}e^{-\pi u^2}$.

Proof. Let $F(x) = e^{-\pi(x+a)^2}$. Since $\int_{-\infty}^{\infty} |e^{-\pi(x+a)^2}|dx < \infty$, we have

$$\begin{aligned}
\hat{F}(u) &= \int_{-\infty}^{\infty} e^{-\pi(x+a)^2}e^{-2\pi ixu}dx \\
&= \int_{-\infty}^{\infty} e^{-\pi[(x+a)^2+2ixu]}dx \\
&= \int_{-\infty}^{\infty} e^{-\pi[(x+a)^2+2ixu-2iau+2iau+u^2-u^2]}dx \\
&= \int_{-\infty}^{\infty} e^{-\pi[(x+a)^2+2iu(x+a)+i^2u^2-2iau+u^2]}dx \\
&= e^{2\pi iau}e^{-\pi u^2}\int_{-\infty}^{\infty} e^{-\pi[(x+a)+iu]^2}dx \\
&= e^{2\pi iau}e^{-\pi u^2}\int_{-\infty}^{\infty} e^{-(\sqrt{\pi}[(x+a)+iu])^2}dx.
\end{aligned}$$

Now let $v = \sqrt{\pi}[(x+a)+iu)]$, so that $dv = \sqrt{\pi}dx$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-[\sqrt{\pi}(x+a)+iu]^2}dx &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-v^2}dv \\
&= \frac{\sqrt{\pi}}{\sqrt{\pi}} = 1,
\end{aligned}$$

so that $\hat{F}(u) = e^{2\pi iau}e^{-\pi u^2}$. □

Example B.3. Let $F(x) = e^{-\pi\left(a+\frac{x}{\sqrt{t}}\right)^2}$. Then $\hat{F}(u) = \sqrt{t}e^{2\pi iau\sqrt{t}}e^{-\pi u^2t}$.

Proof. Let $F(x) = e^{-\pi\left(a+\frac{x}{\sqrt{t}}\right)^2}$. Since $\int_{-\infty}^{\infty} |e^{-\pi\left(a+\frac{x}{\sqrt{t}}\right)^2}|dx < \infty$, we have

$$\begin{aligned}
\hat{F}(u) &= \int_{-\infty}^{\infty} e^{-\pi\left(a+\frac{x}{\sqrt{t}}\right)^2}e^{-2\pi ixu}dx \\
&= \int_{-\infty}^{\infty} e^{-\pi[\left(a+\frac{x}{\sqrt{t}}\right)^2+2ixu+2iau\sqrt{t}-2iau\sqrt{t}+u^2t-u^2t]}dx \\
&= e^{2\pi iau\sqrt{t}}e^{-\pi u^2t}\int_{-\infty}^{\infty} e^{-\pi[\left(a+\frac{x}{\sqrt{t}}\right)^2+2ixu\sqrt{t}\left(a+\frac{x}{\sqrt{t}}\right)+i^2u^2t]}dx \\
&= e^{2\pi iau\sqrt{t}}e^{-\pi u^2t}\int_{-\infty}^{\infty} e^{-\pi[\left(a+\frac{x}{\sqrt{t}}\right)+iu\sqrt{t}]^2}dx \\
&= e^{2\pi iau\sqrt{t}}e^{-\pi u^2t}\int_{-\infty}^{\infty} e^{-[\sqrt{\pi}\left(a+\frac{x}{\sqrt{t}}+iu\sqrt{t}\right)]^2}dx.
\end{aligned}$$

Now let $v = \sqrt{\pi}\left(a + \frac{x}{\sqrt{t}} + iu\sqrt{t}\right)$, so that $dv = \frac{\sqrt{\pi}}{\sqrt{t}}dx$. Then

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-[\sqrt{\pi}\left(a+\frac{x}{\sqrt{t}}+iu\sqrt{t}\right)]^2} dx &= \frac{\sqrt{t}}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-v^2} dv \\ &= \frac{\sqrt{t}\sqrt{\pi}}{\sqrt{\pi}} = \sqrt{t},\end{aligned}$$

so that $\hat{F}(u) = \sqrt{t}e^{2\pi iau\sqrt{t}}e^{-\pi u^2 t}$. □

Theorem B.1. *(Poisson summation formula) Let $F \in L^1(\mathbb{R})$. Suppose that the series*

$$\sum_{m\in\mathbb{Z}} |\hat{F}(m)| < \infty.$$

Then

$$\sum_{n\in\mathbb{Z}} F(n+v) = \sum_{n\in\mathbb{Z}} \hat{F}(n)e^{2\pi inv}.$$

Proof. Let $G(v) = \sum_{\mathbb{Z}} F(n+v)$, so that since $G(v)$ is a function of period 1, we have

$$G(v) = \sum_{\mathbb{Z}} c_n e^{2\pi inv},$$

where the c_n's are the Fourier coefficients,

$$c_n = \int_0^1 G(v)e^{-2\pi inv}dv.$$

Now using our integral we see that

$$\begin{aligned}c_n = \int_0^1 G(v)e^{-2\pi inv}dv &= \int_0^1 \left[\sum_{\mathbb{Z}} F(n+v)\right] e^{-2\pi inv}dv \\ &= \int_0^1 \left[\sum_{\mathbb{Z}} F(n+v)e^{-2\pi inv}\right] dv \\ &= \sum_{\mathbb{Z}} \left[\int_0^1 F(n+v)e^{-2\pi inv}dv\right] \\ &= \int_{-\infty}^{\infty} F(v)e^{-2\pi inv}dv \\ &= \hat{F}(n),\end{aligned}$$

where the second to the last equality is dependent upon the period of v, being 1. Now substituting $\hat{F}(n) = c_n$ into our original equation, we have

$$G(v) = \sum_{\mathbb{Z}} \hat{F}(n)e^{2\pi inv} = \sum_{\mathbb{Z}} F(n+v).$$

□

Example B.4. Using Example B.3, we show

$$\sum_{n\in\mathbb{Z}} F(n) = \sum_{n\in\mathbb{Z}} \hat{F}(n).$$

Proof. Using Theorem B.1, we have

$$\sum_{\mathbb{Z}} \hat{F}(n)e^{2\pi inv} = \sum_{\mathbb{Z}} F(n+v).$$

Letting $v = 0$, we gain

$$\sum_{\mathbb{Z}} \hat{F}(n) = \sum_{\mathbb{Z}} F(n),$$

which is the assertion. □

Example B.5. Using the previous two examples, we have

$$\sum_{n\in\mathbb{Z}} e^{-(n+v)^2\pi/x} = \sqrt{x}\sum_{n\in\mathbb{Z}} e^{-n^2\pi x+2\pi inv}.$$

Proof. Using Theorem B.1, we have

$$\sum_{\mathbb{Z}} \hat{F}(n)e^{2\pi inv} = \sum_{\mathbb{Z}} F(n+v).$$

Starting with

$$\begin{aligned} F(n+v) &= e^{-(n+v)^2\pi/x} \\ &= e^{-\pi\left(0+\frac{n+v}{\sqrt{x}}\right)^2}, \end{aligned}$$

and the equality gained in Example B.3 gives

$$\hat{F}(u) = \sqrt{x}e^{-\pi u^2 x},$$

so that

$$\hat{F}(n) = \sqrt{x}e^{-\pi n^2 x},$$

and again using Theorem B.1, we have our assertion, that;

$$\sum_{n\in\mathbb{Z}} e^{-(n+v)^2\pi/x} = \sqrt{x}\sum_{n\in\mathbb{Z}} e^{-n^2\pi x+2\pi inv}.$$

□

Example B.6. Note that Example B.5 with $v = 0$ is

$$\sum_{n\in\mathbb{Z}} e^{-n^2\pi/x} = \sqrt{x}\sum_{n\in\mathbb{Z}} e^{-n^2\pi x}.$$

Also note that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} e^{-n^2\pi/x} &= \sum_{\mathbb{N}} e^{-n^2\pi/x} + \sum_{\mathbb{N}} e^{-(-n)^2\pi/x} + 1\\ &= 2\sum_{\mathbb{N}} e^{-n^2\pi/x} + 1\\ &= 2\Psi\left(\frac{1}{x}\right) + 1,\end{aligned}$$

where we denote $\sum_{\mathbb{N}} e^{-n^2\pi x} = \Psi(x)$.

Likewise,

$$\begin{aligned}\sum_{n\in\mathbb{Z}} e^{-n^2\pi x} &= \sum_{\mathbb{N}} e^{-n^2\pi x} + \sum_{\mathbb{N}} e^{-(-n)^2\pi x} + 1\\ &= 2\sum_{\mathbb{N}} e^{-n^2\pi x} + 1\\ &= 2\Psi(x) + 1,\end{aligned}$$

so that our equation from Example B.5 becomes

$$2\Psi\left(\frac{1}{x}\right) + 1 = \sqrt{x}[2\Psi(x) + 1],$$

or rather

$$\frac{1 + 2\Psi(x)}{1 + 2\Psi\left(\frac{1}{x}\right)} = \frac{1}{\sqrt{x}},$$

which was used heavily in the section on the Epstein zeta-function.

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